THE SUBGROUP COMPOSED OF THE SUBSTITUTIONS WHICH OMIT A LETTER OF A TRANSITIVE GROUP*

BY

G. A. MILLER

1. Introduction

In a transitive group of order g and of degree n the subgroup G_1 composed of all the substitutions of G which omit a given letter is of order g/n. The properties of G_1 frequently throw light on the possible properties of G. In particular, when G_1 is transitive and of degree $n-1$, G must be multiply transitive, but when G_1 is of a lower degree than $n-1$, there must be at least one substitution besides identity which is commutative with every substitution of G. If the order of such a substitution is less than n its systems of intransitivity are systems of imprimitivity of G. Hence it results that, when G is primitive and G_1 is not of degree $n-1$, G must be the regular group of prime order p.

While a number of such properties relating to G_1 have been known for a long time little has been done towards determining the possible substitution groups which involve a particular G_1. For instance, when G_1 is the symmetric group of degree m, G must be the symmetric group of degree $m+1$, except in the special case when $m = 2$. In this case it may also be the octic group of degree 4, as is well known, but it cannot be any other group. This theorem results directly from the facts that there is no substitution on the letters of the symmetric group of degree $m > 2$ which is commutative with every substitution of this symmetric group and that a regular group of degree m can always be used as the G_1 for at least one transitive group of degree $2m$ and of order $2m^2$. In a similar manner it results that every alternating group, except the alternating group of degree 3, can appear as the G_1 of only the alternating group of the next larger degree, while the alternating group of degree 3 is also the G_1 of a group of order 18 and of degree 6.

As a special case of the theorems just stated there results the well known theorem, first proved by Ruffini, that there is no four-valued rational function on five variables. In fact, if such a function were possible there would be a transitive substitution group of order 30 and of degree 5. The G_1 of this group would have to be the symmetric group of degree 3 since there is no other group of order 6 on less than 5 letters. The fact

* Presented to the Society, December 27, 1924.

9

that there is no three-valued function on five letters follows also from well known properties of G_1. If such a function were possible there would be a transitive group of degree 5 and of order 40. The G_1 of this group would be the octic group of degree 4, since this is the only group of order 8 whose degree is less than 5. If G_1 were this octic group, the cycles of order 2 in G would involve more than 40 letters and hence G would involve substitutions of degree 5 containing cycles of order 2. This is impossible since such substitutions would be of order 6, and 6 does not divide 40. A general theorem which covers this case may be stated as follows:

If cycles which appear in as many sets of conjugates under G as under G_1 are contained in substitutions whose degree is less than $n-1$, then cycles of this order appear also in substitutions of degree n contained in G.

From the fact that the number of letters omitted by G_1 must divide the degree of G it results directly that a group of degree m cannot appear as the G_1 of a group unless the degree of this group is one of the following m numbers: $m+1,\ m+2, \cdots, 2m$. In particular, the number of the different transitive substitution groups which involve the same substitution group as a subgroup composed of all the substitutions which omit one letter is always finite. In fact, if this substitution group is of degree m the number of these transitive groups is clearly less than the number of the possible transitive groups on $m+1,\ m+2, \cdots, 2m$ letters. As the number of the possible substitution groups on a given number of letters is finite our theorem is obviously true. From the fact that the number of letters omitted by all the substitutions of G which omit one letter is a divisor of the degree of G_1, as well as of the degree of G, it results that the only group of degree m which is a G_1 of at least one group of each of the m degrees $m+1,\ m+2, \cdots, 2m$ is the group of order and of degree 2. From the same fact it results also that a group of prime degree p cannot be the G_1 of any group unless the degree of this group is either $p+1$ or $2p$. In the latter case this group of prime degree must also be regular.

2. The subgroup G_1 has a prime order

At the close of the preceding section it was noted that when G_1 is of a prime degree p it cannot appear in any transitive group unless the degree of G is either $p+1$ or $2p$. In the latter case G is imprimitive and G_1 is regular. There is obviously one, and only one, such group for every prime number p. Its order is $2p^2$ and it can be constructed by extending by means of a substitution of order 2 and of degree $2p$ the direct product of two regular groups of order p. Hence all the transitive substitution groups which involve as a G_1 a group of prime degree p but have themselves a degree which exceeds $p+1$ are completely determined.

When G_1 is regular and of degree p the group G of degree $p+1$ is clearly impossible unless $p+1$ is either 3 or of the form 2^α. In the latter case G contains the abelian group of order 2^α and of type $(1, 1, 1, \cdots)$ as a characteristic subgroup, while all of the remaining substitutions are of order and degree p. Since the order of the group of isomorphisms of this abelian group is not divisible by p^2 it results that all of its subgroups of order p are conjugate, and hence there is only one such group of order $p \cdot 2^\alpha$. This completes a proof of the following theorem:

The regular group of odd prime order p is always the G_1 of one and only one transitive group of degree $2p$. In the special case when p is of the form $2^\alpha - 1$ it is also the G_1 of one and only one group of degree $p+1$. It cannot be the G_1 of any other transitive group.

The regular group of order 2 is obviously the G_1 of the octic group and also of the symmetric group of degree 3.

The developments which precede clearly constitute a special case of those which relate to the groups whose G_1 is of prime order p but involves $k > 1$ cycles. When the corresponding G is of degree $kp+1$ it is well known that it involves a characteristic regular group of order $kp+1$ and that all its remaining substitutions are regular and of degree kp. At least one such group is evidently always possible when $kp+1$ is of the form 2^α. It is also always possible when $p = 2$ since any abelian regular group of order $2k+1$ may be extended by a substitution of order 2 and of degree $2k$ which transforms into their inverses all the substitutions of this regular group. As these are the only possible groups of degree $2k+1$ when G_1 is generated by a substitution of order 2 and of degree $2k$ it results that *the number of the transitive groups of degree $2k+1$ which have for their G_1 the group of order 2 and of degree $2k$ is exactly the same as the number of the abstract abelian groups of order $2k+1$, k being an arbitrary positive integer.*

When G_1 is of order 2 and of degree 2^α the degree of G is of the form $2^\alpha + 2^\beta$ where $\beta \leq \alpha$, according to a theorem noted above. It is not difficult to prove that β can have every integral value from 0 to α. As β can have no other value according to the general theorem to which we have just referred it results that when G_1 is of order 2 and of degree 2^α the possible degrees of G are completely known. To prove that β can have every value from 0 to α it is only necessary to note that when β has any one of these values it is possible to construct a regular abelian group of order $2^\alpha + 2^\beta$ which involves a subgroup of order 2^β and of type $(1, 1, 1, \cdots)$ while the order of the remaining substitutions exceeds 2. Hence there is a substitution of order 2 and of degree 2^α which transforms every substitution of this regular group into its inverse.

In fact, the number of the substitutions which have this property is obviously $2^{\alpha-\beta}+1$. In the special case when $\beta=0$ we thus obtain one of the generalized dihedral groups noted at the close of the next to the preceding paragraph. While all the possible groups are known in this case this is not true in general. In fact, when $\beta>0$ it is sometimes possible to extend a non-abelian regular group so as to obtain one of the groups in question, as may be seen from the fact that when $\alpha=3$ and $\beta=2$ the regular tetrahedral group may be extended by means of a substitution of order 2 and of degree 2^α so as to obtain a group whose G_1 is of order 2 and of degree 2^α.

The preceding developments exhibit the interesting fact that G_1 can be so chosen that the number of the different possible degrees of the corresponding transitive groups exceeds any given number, since when G_1 is of order 2 and of degree 2^α the number of such possible degrees is exactly $\alpha+1$, where α is an arbitrary positive integer. Hence the theorem noted in the Introduction of the present article, which states that a given G_1 can appear in only a finite number of transitive groups as the subgroup composed of all the substitutions which omit a given letter, does not imply that there is a number which cannot be exceeded by the number of the different transitive groups which involve the same group for their G_1. In fact, such a number does not exist, according to the theorem just proved.

We noted above that when β has one extreme value, viz., 0, the possible groups are well known. When β has the other extreme value, viz., α, it is easy to prove that G involves a subgroup of half its order which can be constructed by establishing a (2,2) isomorphism between two regular groups of order $2^{\alpha-1}$. To prove this theorem it may first be noted that, when β has this maximal value G_1 has two conjugates under G, and hence it must be transformed into itself by exactly one-half of the substitutions of G. These substitutions constitute a subgroup of one-half of the order of G and they include the two conjugates of G_1 since these conjugates involve no common letter and are therefore commutative. Since all of the remaining substitutions of G are of degree $2^{\alpha+1}$ this subgroup must be intransitive and involve two systems of intransitivity.

The fact that the two transitive constituents of this intransitive subgroup are regular follows from the degree and order of G_1. It results, in particular, that G involves at least one set of systems of imprimitivity whose constituents are of degree 2^α, as well as one whose constituents are of degree 2. Moreover, it is evident that if we can establish a (2,2) isomorphism between any two conjugate regular groups such that the group thus formed can be extended by a substitution which interchanges its systems of intransitivity, transforms it into itself, and has its square in it, we obtain a group

whose G_1 is of order 2 and has two conjugates under the group. It should also be noted that the G_1 of every non-regular group of degree n has the property that it and any one of its conjugates which differs from it generates a group of degree n, for if this were not the case the group which they generate would be contained in G_1. Hence the theorem:

Every two different subgroups which are separately composed of all the substitutions which omit one letter of a transitive group of degree n must generate a group of degree n.

3. THE SUBGROUP G_1 IS TRANSITIVE

When G_1 is a transitive group of degree m whose subgroup composed of all the substitutions which omit one letter is of degree $m-r$, then the degree of G cannot exceed $m+r$. In fact, this degree is $m+k$, where k is a divisor of r, as may be seen directly from the fact that the k substitutions which are commutative with every substitution of G must be commutative with every substitution of G_1, but the substitutions which have the latter property and involve only the letters of G_1 constitute a group of order r whose transitive constituents are known to be regular groups. In particular, it results from these considerations that a necessary and sufficient condition that a transitive group of degree m can appear as the G_1 of a transitive group of degree $2m$ is that the former group be regular.

From the preceding paragraph it results directly that, when G_1 is a non-regular primitive group of degree m, G must be a primitive group of degree $m+1$. It is not always possible to construct such a primitive group of degree $m+1$ when G_1 is given. It was noted in the Introduction that, when G_1 is non-abelian and either alternating or symmetric, there is one and only one transitive group of degree $m+1$ whose subgroup composed of all its substitutions which omit one letter constitutes this G_1. Hence it is easy to verify that 6 is the smallest value of m such that a primitive group of degree m cannot be used as the G_1 of any transitive group whatever. In fact, neither of the two well known primitive groups of degree 6 and of orders 60 and 120 respectively can appear as the G_1 of a group of degree 7, since the cycles of order 7 would all have to be conjugate if the group were of order 840 and hence the number of the subgroups of order 7 would be 20, which is incongruent to unity modulo 7, and if the group were of order 420 there would be at most three sets of conjugate cycles of order 7, but the number of subgroups of order 7 could not be 10, 20, or 30.

It was noted in the preceding section that when G_1 is of order 2 and of degree 2^α it appears in transitive groups of exactly $\alpha+1$ different degrees. When G_1 is transitive and of degree 2^α it obviously appears also in transitive groups of $\alpha+1$ different degrees when α is 1 or 2, but when α is 3 this

is not the case. In fact, a transitive group of degree 2^α can appear in transitive groups of $\alpha+1$ different degrees only when the group of degree 2^α is regular, since all such degrees must be of the form $2^\alpha+2^\beta$ where $\beta<\alpha+1$, and there is no transitive group of degree 10 whose G_1 is a regular group of degree 8. This fact results directly from Sylow's theorem, since a group of order 80 contains either 1 or 16 subgroups of order 5. If it contains only one such subgroup it must involve an abelian group of order 20, but an abelian group of this order cannot be represented on ten letters. If it contains 16 such subgroups it contains also an invariant subgroup of order 16 and this must involve five systems of intransitivity when it is represented on ten letters. Hence it could not involve a regular subgroup of order 8.

While it results from the preceding paragraph that for at least one value of α it is impossible to construct transitive groups of as many as $\alpha+1$ different degrees whose G_1 is a transitive group of degree 2^α it is easy to prove that it is always possible to construct a transitive group of degree $2^\alpha+2^{\alpha-1}$ whose G_1 is a transitive group of degree 2^α. In fact, to construct such a group in which G_1 is the regular abelian group of order 2^α and of type $(1, 1, 1, \cdots)$ we may first construct two abelian groups of order $2^{\alpha-1}$ and of degree 2^α whose two transitive constituents are regular groups of type $(1, 1, 1, \cdots)$ such that these two abelian groups have one transitive constituent in common. The direct product of these two groups is of degree $2^\alpha+2^{\alpha-1}$ and its three transitive constituents can be transformed according to the symmetric group of degree 3 so as to obtain two transitive groups of order $3\cdot 2^\alpha$ whose three Sylow subgroups of order 2^α are abelian and regular. Hence it results that *for every value of $\alpha>1$ it is possible to construct two transitive groups of degree $2^\alpha+2^{\alpha-1}$ which have for their G_1 an abelian regular group of order 2^α*. It may be noted that the two transitive groups of degree 6 which are simply isomorphic with the symmetric group of degree 4 are illustrations of this general theorem.

It results from the preceding paragraph that there are always transitive groups of at least two different degrees which involve for their common G_1 a certain regular group of order 2^α but it is not known whether it is possible to construct a transitive group which appears as the common G_1 of transitive groups of a number of different degrees exceeding an arbitrarily large number. Hence we do not have here a theorem which corresponds to the theorem established in the preceding section as regards transitive groups which have for their common G_1 an intransitive group of order 2. A necessary and sufficient condition that some regular group of degree m is the G_1 of a transitive group of degree $m+1$ is that $m+1$ is a power of a prime number. If there is only one such regular group it is well known to be cyclic.

When G_1 is a transitive group of degree m and G is of degree $m+k$,

$k > 1$, then G must transform $m/k+1$ systems of imprimitivity according to a multiply transitive group. In particular, when G_1 is a regular group which transforms each of its systems of imprimitivity, involving k letters in a set, according to a regular group, then $m/k+1$ is either a prime number or a power of some prime number. This is obviously always the case when G_1 is abelian. From the fact that the systems of imprimitivity of such a group are transformed according to a multiply transitive group whose class is one less than its degree it results that *a regular abelian group of order m cannot appear as the G_1 of a transitive group of degree $m+k$ except when $m/k+1$ is either a prime number or a power of such a number.*

It was noted above that the cyclic regular group of degree 4 is the G_1 of a transitive group of degree 6. It is not difficult to prove that the cyclic regular group of order m cannot be the G_1 of a transitive group of degree $m+2$ whenever $m > 4$. Such a group G would contain $m/2+1$ conjugates of G_1 and hence each of these conjugates would be transformed into itself by exactly $2m$ substitutions of G. Each of these conjugates would be transformed into itself by exactly two substitutions found in each of the others. Hence all the substitutions of order 2 found in these conjugates would be commutative and would generate an abelian group which would transform each of these conjugates into itself. As this abelian group could not have more than two substitutions in common with one of these conjugates its order could not exceed 4. Hence $m/2+1$ could not exceed 3. That is, m could not exceed 4, which is in accord with the statement made above. This proof evidently applies also to all other regular groups which contain only one subgroup of order 2.

By a method which is somewhat similar to the one employed in the preceding paragraph it is easy to prove that many other regular groups of order m cannot appear as the G_1 of some transitive group of degree $m+2$. In particular, when m is of the form 2^{α} it is well known that every possible non-cyclic group of order 2^{α} with the exception of 3 groups contain an invariant non-cyclic subgroup of order 4 whenever $\alpha > 3$.* We proceed to prove that when a group of order 2^{α} which involves a non-cyclic invariant subgroup of order 4 is represented as a regular group then its group of isomorphisms cannot involve a substitution of order 2 and of degree $2^{\alpha}-2$. If such a substitution s could exist it would have to transform two of the substitutions of order 2 contained in the given invariant subgroup of order 4 into themselves multiplied by the third of its substitutions of order 2. The substitution s would also have to transform a substitution not found in the given non-cyclic subgroup of order 4 into

* G. A. Miller, these Transactions, vol. 6 (1905).

itself multiplied by one of the two substitutions of this subgroup with which it is not commutative. From this fact it follows directly that s^2 could not be identity, and hence s could not be of order 2 as was assumed.

From the preceding two paragraphs, it results that if a regular group of order 2^{α} appears as the G_1 of a transitive group of degree $2^{\alpha}+2$, $\alpha > 2$, it cannot be either cyclic or dicyclic, and it cannot contain a non-cyclic invariant subgroup of order 4. Hence it must be one of two groups, viz., the dihedral group or the group of order 2^{α} which involves operators of order $2^{\alpha-1}$ and $2^{\alpha-2}+1$ operators of order 2. The latter must be excluded since its group of isomorphisms cannot involve a substitution of order 2 and degree $2^{\alpha}-2$ when this group is represented as a regular substitution group. It remains therefore only to consider whether the regular dihedral group of order 2^{α}, $\alpha > 3$, can be the G_1 of a transitive group of degree $2^{\alpha}+2$. We proceed to prove that this is impossible.

Suppose that there existed a transitive group G of degree $2^{\alpha}+2$ whose G_1 is the regular dihedral group of order 2^{α}, $\alpha > 3$. The subgroup G_1 must be transformed into itself by $2^{\alpha+1}$ substitutions of G. These substitutions constitute an intransitive group whose two transitive constituents are of degree 2 and 2^{α} respectively. The transitive constituent of degree 2^{α} must involve a substitution s of order 2 and of degree $2^{\alpha}-2$ which is commutative with 2 and only 2 of the substitutions of G_1. It is well known that the group of isomorphisms of G_1 is simply isomorphic with the holomorph of the cyclic group of order $2^{\alpha-1}$.[*] Hence this group of isomorphisms involves $2^{\alpha-1}+2^{\alpha-2}+2+1$ operators of order 2. The given substitution s must therefore transform the substitutions of order $2^{\alpha-1}$ contained in G_1 into their inverses while it transforms the non-invariant substitutions of order 2 in G_1 into themselves multiplied by operators of order $2^{\alpha-1}$. It results from this property of s that the group generated by G_1 and s is the dihedral group of order $2^{\alpha}+1$. Hence G must involve this dihedral group, represented as an intransitive group having transitive constituents of degrees 2 and 2^{α} respectively.

From the preceding paragraph it results that the substitution of order 2 which is commutative with every substitution of G must have for its constituent of degree 2^{α} the substitution of order 2 generated by a substitution of order 2^{α} found in the given dihedral group of order $2^{\alpha+1}$. There would be $2^{\alpha-1}+1$ conjugates of this dihedral group in G. Two of these conjugates would have in common $2^{\alpha}-2$ letters and hence the substitutions of order 2 which are commutative with all the substitutions found in these two conjugates and involve only their letters would have more than one

---

[*] Miller, Blichfeldt, Dickson, *Theory and Applications of Finite Groups*, 1916, p. 169.

cycle in common since they could involve only cycles which are found in the substitution of order 2 which is commutative with every substitution of G. As this is impossible we have established the following general theorem:

A regular group of order 2^α, $\alpha > 2$, cannot appear as the subgroup composed of all the substitutions which omit one letter of a transitive group of degree $2^\alpha+2$.

The theorem noted above that a regular cyclic group of order m cannot be the G_1 of a transitive group of degree $m+2$ whenever $m>4$ is obviously a special case of the following theorem:

If a regular abelian group of order m is the G_1 of a transitive group of degree $m+k$, then $m \leqq 2k$.

To prove this theorem it may first be noted that every two different conjugates of G_1 must involve all the letters of this transitive group G of degree $m+k$. The k substitutions which are commutative with every substitution of G must therefore have constituents of degree m in common with substitutions of these two conjugates of G_1 respectively. Hence these conjugates cannot have more than k letters in common. If they have k letters in common it results that $2m-k=m+k$, and hence $m=2k$. If they have less than k letters in common m is evidently less than $2k$. Hence the theorem under consideration has been established. The fact that this theorem does not hold for regular non-abelian groups results directly from the group of isomorphisms of the non-cyclic group of order 9, since this is a transitive group of degree 8 whose G_1 is the non-cyclic regular group of order 6.

At the close of the preceding section it was noted that every two distinct conjugates of a subgroup composed of all the substitutions of a transitive group which omit a given letter generate a group whose degree is equal to the degree of the entire group. A special case of this theorem is that each such subgroup must involve at least half of the letters of the entire transitive group. When two such subgroups generate a transitive group it must evidently be the entire group since the subgroup composed of all the substitutions which omit one letter of this transitive group is the same as the subgroup of the entire transitive group and the degrees of these transitive groups are the same. In particular, it results that whenever the subgroup composed of all the substitutions which omit one letter of a transitive group is transitive then this subgroup and any of its conjugates which differ from it generate the entire group whenever the degree of this subgroup exceeds one-half the degree of the group. In particular, a necessary and sufficient condition that two distinct conjugates of a subgroup composed of all the substitutions which omit one letter of a transitive group generate the entire group is that they generate a transitive group.

UNIVERSITY OF ILLINOIS,
URBANA, ILL.

ON THE CLOSENESS OF APPROACH OF COMPLEX RATIONAL FRACTIONS TO A COMPLEX IRRATIONAL NUMBER*

BY

LESTER R. FORD

Dirichlet showed that if ω is a real irrational, the inequality

$$\left| \omega - \frac{p}{q} \right| < \frac{k}{q^2}$$

is satisfied by an infinite number of real rational fractions, p/q, when $k = 1$. Later Hermite gave a method, based on binary quadratic forms, of constructing an infinite suite of fractions satisfying the inequality when $k = 1/\sqrt{3}$. The problem of the minimum value of k was solved by Hurwitz†, who showed that if $k = 1/\sqrt{5}$ an infinite number of fractions satisfy the inequality, whereas if $k < 1/\sqrt{5}$ there is an ω in every interval of the real axis for which the inequality holds for only a finite number of rational fractions. Proofs of Hurwitz' theorem have been given by Borel‡, by Humbert§ and by the present author‖.

In the present paper we propose to investigate the analogous problem in the complex domain. Let ω be any complex irrational number and consider the inequality

$$(1) \qquad \left| \omega - \frac{p}{q} \right| < \frac{k}{q\bar{q}},$$

where k is real, p/q is a complex rational fraction (i. e., p and q are each of the form $m + i\,n$, where m and n are real integers), and $\bar{q}$ is the conjugate imaginary of q.

Hermite¶ has demonstrated, again by the use of quadratic forms, the existence of an infinite suite satisfying the inequality when $k = 1/\sqrt{2}$. However, the problem of the least value of k, such that for any ω there

* Presented to the Society, April 24, 1920, under the title *A theorem relative to rational approximations to irrational complex numbers.*

† Mathematische Annalen, vol. 39 (1891), pp. 279-284.

‡ Journal de Mathématiques, ser. 5, vol. 9 (1903), pp. 329 ff.

§ Journal de Mathématiques, ser. 7, vol. 2 (1916), pp. 155-167.

‖ Proceedings of the Edinburgh Mathematical Society, vol. 35, Part 2, Session 1916-1917.

¶ Journal für Mathematik, vol. 47 (1854), pp. 357-366. Hurwitz, in Acta Mathematica, vol. 11 (1887), pp. 187-200, gives an infinite suite satisfying the inequality when $k = 1$.

is always an infinite number of fractions satisfying the inequality, has not been solved hitherto. The difficulty has been that the early methods used in the real case, those of Hurwitz and Borel, employ continued fractions and make use of properties not possessed by any satisfactory extension of ordinary continued fractions to the case of complex numbers. The geometric methods used by the author are not subject to this limitation.

We shall prove the following

THEOREM. *If $k = 1/\sqrt{3}$ there is an infinite number of rational fractions satisfying the inequality*

$$(1) \qquad \left| \omega - \frac{p}{q} \right| < \frac{k}{q\bar{q}}.$$

If $k < 1/\sqrt{3}$ there exists a set of irrational numbers, everywhere dense in the complex plane, for each of which the inequality (1) *is satisfied by only a finite number of rational fractions.*

The geometric problem. L-lines and S-spheres. Visualizing the Argand diagram on which the complex numbers are represented as a horizontal plane, we shall be concerned with geometrical constructions in the three-dimensional space lying on one side of this plane, say above it. Through the point ω under consideration let a line, L, be drawn perpendicular to the complex plane. At each rational point, p/q (in its lowest terms), let a sphere, S, be constructed, tangent to the complex plane at that point, of radius $1/2hq\bar{q}$, and lying in the upper half-space.

If L intersects the S-sphere corresponding to p/q the distance between ω and p/q is less than the radius, or

$$(2) \qquad \left| \omega - \frac{p}{q} \right| < \frac{1}{2hq\bar{q}},$$

otherwise the inequality does not hold. We must show then that when $h = \frac{1}{2}\sqrt{3}$, L intersects an infinite number of these spheres and that when $h > \frac{1}{2}\sqrt{3}$ the L-lines constructed through certain of the irrationals intersect only a finite number of spheres.*

The group of Picard. The spheres which we have just defined are connected with the group of linear transformations

$$(3) \qquad z' = \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

* It is clearly unnecessary to consider fractions not in their lowest terms. If p/q fails to satisfy an inequality of the type (2) an equal fraction with a larger absolute q will fail to satisfy it; if p/q satisfies the inequality only a finite number of equal fractions will satisfy it.

where a, b, c, d are complex integers, in the following manner. If the transformations of the group be defined as space transformations according to the method of Poincaré* the upper half-space is transformed into itself. If we add to the spheres of the preceding section the plane $\zeta = h$ (ζ being measured along an axis through the origin perpendicular to the complex plane) the resulting set of spheres is invariant under the transformations of the group.† The plane $\zeta = h$ is the S-sphere of the point ∞.

Division of the half-space into pentahedra. Of prime importance in the study of the group is the fundamental pentahedron, discovered by Bianchi.‡ Write $z = \xi + i\eta$. The fundamental pentahedron is the portion of space lying above the unit sphere with center at the origin, $\xi^2 + \eta^2 + \zeta^2 = 1$, and bounded by the four planes $\xi = \pm\frac{1}{2}$, $\eta = \pm\frac{1}{2}$. If this solid be inverted in its faces and each new pentahedron be inverted in its faces, and so on ad infinitum, the whole upper half-space is filled up without overlapping. This set of pentahedra has the property of invariance; that is, each transformation of the group carries each pentahedron into some other. Furthermore, there exists a transformation carrying a given pentahedron into a specified one; in particular any one can be carried into the fundamental pentahedron by a suitable transformation.

The invariant set of pentahedra and the invariant set of S-spheres will play a fundamental part in the proof. One further fact will be used: The segment of an L-line bounded by a point in the upper half-space and by the irrational point ω in the complex plane intersects an infinite number of pentahedra.

Proof of the first part of the theorem. Let $h = \frac{1}{2}\sqrt{3}$. Let us suppose that for a given irrational ω the corresponding L-line intersects only a finite number of S-spheres. Then, in the neighborhood of ω, L passes successively through an infinite number of pentahedra and remains exterior to, or at most tangent to, all S-spheres. We shall show that this is impossible.

Let D be a pentahedron through which L passes. Make a transformation of the group of Picard carrying D into the fundamental pentahedron. Then, since circles are carried into circles and angles are preserved, L is carried into a semi-circle orthogonal to the complex plane and passing through the fundamental pentahedron.

---

* Acta Mathematica, vol. 3 (1884), pp. 49–92. Poincaré bases his extension on the well known fact that a linear transformation in the complex plane is equivalent to an even number of inversions in circles. By making inversions in spheres having these circles as equators a space transformation results. Certain properties are immediate: the transformations are conformal; spheres are carried into spheres; and circles are carried into circles.

† For a full discussion of the geometry of the group of Picard see a paper by the present author, these Transactions, vol. 19 (1918), pp. 1–42.

‡ Mathematische Annalen, vol. 38 (1891), pp. 313–333.

We shall now investigate whether we can so place a semi-circle, C, that it shall be orthogonal to the z-plane, that it shall pass through the fundamental pentahedron, and that it shall intersect none of the S-spheres that lie in the neighborhood. The points of the fundamental pentahedron nearest the z-plane (the vertices) have the ζ-coördinate $\frac{1}{2}\sqrt{2}$. This fact, together with the requirement that C shall not penetrate the region above the plane $\zeta = \frac{1}{2}\sqrt{3}$, gives for the radius, r, of C the following bounds:

$$\frac{1}{2}\sqrt{2} \leqq r \leqq \frac{1}{2}\sqrt{3}.$$

Consider now the S-spheres of the integral points. These are tangent to the complex plane at the integral points and have the radius $1/\sqrt{3}$. They are cut by a plane $\zeta =$ const. in a set of circles whose centers lie vertically above the integral points. Consider the S-sphere of the point $0/1$; viz.,

$$\xi^2 + \eta^2 + \zeta^2 - \frac{2}{\sqrt{3}}\zeta = 0.$$

Setting $\zeta = \frac{1}{2}\sqrt{3}$ and again $\zeta = \sqrt{3}/6$ in this equation we have in each case

$$\xi^2 + \eta^2 = \frac{1}{4}.$$

Since the radius is $\frac{1}{2}$ we see that each of these planes cuts the S-spheres of the integral points in a set of tangent circles [Fig. 1]. Horizontal planes lying between these two planes cut the spheres in larger circles.

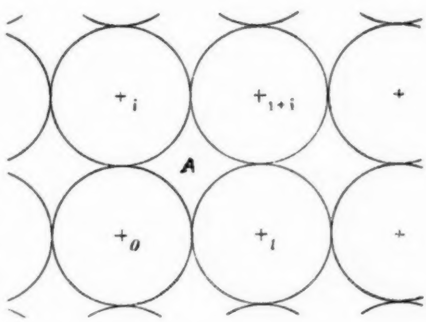

Fig. 1.

Let K_1, K_2 be the intersections of C with the plane $\zeta = \sqrt{3}/6$. If C is to remain exterior to the spheres in question, K_1 and K_2 must lie in the regions, such as A in Fig. 1, exterior to the circles. Now K_1 and K_2 cannot lie in the same region A; for the greatest length that can be laid down in A is 1, whereas the chord $K_1 K_2$ at a distance $\sqrt{3}/6$ from the center of a circle of radius not less than $\frac{1}{2}\sqrt{2}$ is easily found to be not less than $\sqrt{15}/3$, or 1.29.

There remains the possibility that K_1 and K_2 lie in different regions, the arc $K_1 K_2$ passing above certain of the S-spheres under consideration. Now considering Fig. 1 as the intersection of the plane $\zeta = \frac{1}{2}\sqrt{3}$ with

the S-spheres, we see that adjacent S-spheres have a common horizontal tangent line lying in the plane $\zeta = \frac{1}{2}\sqrt{3}$ at the points where the circles of intersection are tangent. The only position which C can have in order not to penetrate into the interior of one of these spheres or pass above the plane $\zeta = \frac{1}{2}\sqrt{3}$ is to touch two spheres and the plane at the point where they have a common horizontal tangent.

The foregoing restricts C to one of four positions in the faces of the fundamental pentahedron. On account of symmetry it will suffice to consider one case: C lies in the plane $\xi = \frac{1}{2}$, has the radius $\frac{1}{2}\sqrt{2}$, and the center $\xi = \frac{1}{2}$, $\eta = \zeta = 0$.

C meets the z-plane in the points

$$z_1 = \frac{1}{2} + \frac{1}{2} i\sqrt{3}, \qquad z_2 = \frac{1}{2} - \frac{1}{2} i\sqrt{3}.$$

Now both these points are irrational, whereas the transform of an L-line meets the z-plane in one rational point; namely, the transform of ∞. For if $z = \infty$ in (3) we have $z' = a/c$, a rational. Consequently this semicircle cannot be the transform of an L-line.

We have proved that every transform of an L-line which passes through the fundamental pentahedron has a segment either above the plane $\zeta = \frac{1}{2}\sqrt{3}$ or within an S-sphere corresponding to an integral point, and furthermore that this enclosed segment lies above the plane $\zeta = \sqrt{3}/6$. We can construct a region above the latter plane within which we are certain that such a segment lies; for instance, the region above the plane $\zeta = \sqrt{3}/6$ and enclosed by the cylinder $\xi^2 + \eta^2 = 25$. Let N be the number of pentahedra extending within this region. It is known from the geometry of the pentahedral division that N is finite. The number of pentahedra in the region through which the semicircular transform of L passes is less than N.

Carrying these results back to the original pentahedron D we can state that of some N successive pentahedra, including D, through which L passes, there is a segment of L lying within an S-sphere. It follows that L cannot pass successively through an infinite number of pentahedra and remain exterior to, or at most tangent to, all S-spheres. This proves the first part of the theorem.

Proof of the second part of the theorem. Consider the semi-circle C just discussed. We shall show that for its terminus, $\omega = \frac{1}{2} + \frac{1}{2} i\sqrt{3}$, the inequality (2) holds for only a finite number of rational fractions when $h > \frac{1}{2}\sqrt{3}$.

The proof will require a careful study of the situation of C with reference to the S-spheres when $h = \frac{1}{2}\sqrt{3}$. We shall show that, with this value of h, C is tangent to an infinite number of S-spheres but penetrates into the interior of none.

The irrationals $\frac{1}{2}+\frac{1}{2}i\sqrt{3}$ and $\frac{1}{2}-\frac{1}{2}i\sqrt{3}$ are the roots of the equation

$$z^2-z+1=0.$$

This equation can be written in the form

$$z=\frac{(2-i)z+2i}{-2iz+2+i},$$

which shows that z_1 and z_2 are the fixed points of the transformation

$$T: \qquad z'=\frac{(2-i)z+2i}{-2iz+2+i}.$$

Since

$$ad-bc=(2-i)(2+i)+(2i)^2=1,$$

it follows that T is a transformation of the group of Picard.

Since $a+d=4$ the transformation is of the type known as hyperbolic (the condition being that $a+d$ be real and $|a+d|>2$). It is characteristic of this type of transformation that all circular arcs joining fixed points are invariant under the transformation. C is such an invariant arc.

Furthermore, if P is a point on C and P' is its transform when the transformation T is made, then by repeating the transformation T and its inverse the transforms of the arc PP' cover the semi-circle C completely without overlapping. We shall presently choose a convenient point P, after which it will suffice to determine the situation of the arc PP' with reference to S-spheres in order to know the situation of the whole semi-circle.

We shall begin by considering the S-spheres of the numbers $-i$ and $1/(1+i)$. Their equations are

$$S_1(-i): \qquad \xi^2+(\eta+1)^2+\left(\zeta-\frac{1}{\sqrt{3}}\right)^2=\left(\frac{1}{\sqrt{3}}\right)^2,$$

$$S_2\left(\frac{1}{1+i}\right): \qquad \left(\xi-\frac{1}{2}\right)^2+\left(\eta+\frac{1}{2}\right)^2+\left(\zeta-\frac{1}{2\sqrt{3}}\right)^2=\left(\frac{1}{2\sqrt{3}}\right)^2.$$

The intersections of these spheres with the plane $\xi=\frac{1}{2}$, in which C lies, are the circles

$$C_1: \qquad (\eta+1)^2+\left(\zeta-\frac{1}{\sqrt{3}}\right)^2=\frac{1}{12},$$

$$C_2: \qquad \left(\eta+\frac{1}{2}\right)^2+\left(\zeta-\frac{1}{2\sqrt{3}}\right)^2=\frac{1}{12}.$$

These circles in the plane $\xi=\frac{1}{2}$ are shown drawn to scale in Fig. 2.

We shall now show that C is tangent to these circles. The equation of C in the plane $\xi = \frac{1}{2}$ is

$$C: \qquad \eta^2 + \zeta^2 = \left(\frac{1}{2}\sqrt{3}\right)^2 = \frac{3}{4}.$$

From the proportionality of the coördinates of the centers of C_1 and C_2,

$$-1 : \frac{1}{\sqrt{3}} = -\frac{1}{2} : \frac{1}{2\sqrt{3}},$$

we see that the centers lie on a line,

$$\eta = -\sqrt{3}\,\zeta,$$

through the center $(0, 0)$ of C. This line cuts C in a point, P, whose coördinates are found to be $\eta = -\frac{3}{4}$, $\zeta = \frac{1}{4}\sqrt{3}$. We verify easily that P lies on both C_1 and C_2. Since the circles pass through a point on their line of centers they are mutually tangent at the point.

We note further that the S-sphere of the point $1 - i$ cuts the plane $\xi = \frac{1}{2}$ in C_1, since the sphere is the reflection of S_1 in that plane. We have shown then that C, which was constructed so as to be tangent to the S-spheres of 0, 1, and ∞, is tangent also to the S-spheres of $-i$, $1 - i$, and $1/(1+i)$.

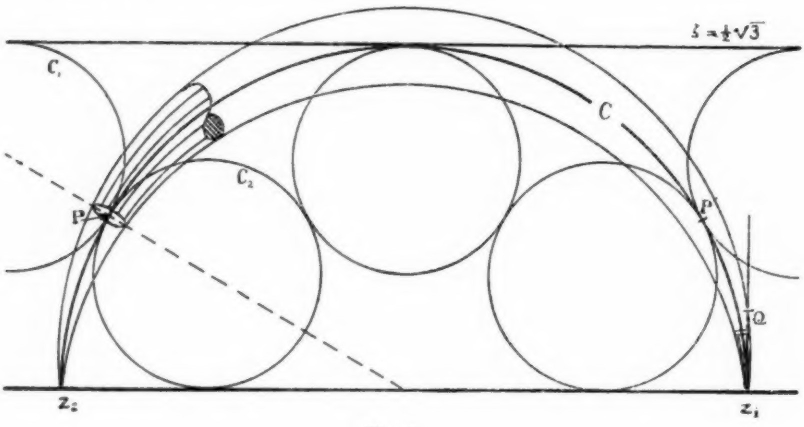

Fig. 2.

If now we make a reflection in the vertical plane through the real axis, $\eta = 0$, with respect to which the system of pentahedra, the system

of S-spheres, and the semi-circle C are symmetrical, the spheres corresponding to the points $-i$, $1-i$, and $1/(1+i)$ are carried into the spheres corresponding to i, $1+i$, and $1/(1-i)$ respectively. Hence the spheres corresponding to these latter points touch C at a point P', whose coördinates in the plane $\xi=\frac{1}{2}$ are $\eta=\frac{3}{4}$, $\zeta=\frac{1}{4}\sqrt{3}$.

Now let us make the transformation T. The points $-i$, $1-i$, and $1/(1+i)$ are carried, as we find by substituting in the equation of the transformation, into the points i, $1+i$, and $1/(1-i)$ respectively. Since S-spheres go into S-spheres and since C is invariant, it follows that P is transformed into the point common to C and to the S-spheres of these latter three points; that is, into P'.

The nine S-spheres tangent to C along the arc PP' are transformed by repetitions of T and its inverse into an infinite number of S-spheres tangent to C.

In order to prove that no point of C is interior to an S-sphere it suffices to prove that no point of PP' is interior to an S-sphere. Now in order that an S-sphere should contain a point of PP' it is necessary that its diameter be greater than the ζ-coördinate of P; that is,

$$\frac{2}{\sqrt{3}\,q\bar{q}} > \frac{1}{4}\sqrt{3}, \quad \text{or} \quad q\bar{q} < \frac{8}{3}.$$

Now $q\bar{q} \geqq 4$ except for the values $q=1$ and $q=1+i$ (or these values multiplied by -1 or $\pm i$). The former are the integral points; the latter are the points $m+ni+\frac{1}{2}(1+i)$.

Again an S-sphere cannot contain any point of C unless the distance from the rational point to which the sphere belongs to the nearest point of the segment $z_1 z_2$ is less than the radius. The radii of the two classes of spheres just mentioned are $1/\sqrt{3}$ and $\sqrt{3}/6$. There are no points of these two classes, other than $0, 1, i, -i, 1+i, 1-i, \frac{1}{2}(1+i)$, and $\frac{1}{2}(1-i)$, which have already been considered, lying within the larger of these two distances. The nearest is $\frac{1}{2}(1+3i)$ (or the similarly situated point $\frac{1}{2}(1-3i)$) whose distance from z_1 is $|\frac{1}{2}(1+3i)-\frac{1}{2}(1+i\sqrt{3})|$, or $\frac{1}{2}(3-\sqrt{3})$, which is greater than $1/\sqrt{3}$.

We have shown that the arc PP' is tangent to certain S-spheres but penetrates to the interior of none. It follows that the whole semi-circle C has no points interior to an S-sphere but that it touches an infinite number of S-spheres.

Now let h be greater than $\frac{1}{2}\sqrt{3}$. Then the plane $\zeta=h$ lies above the plane $\zeta=\frac{1}{2}\sqrt{3}$, and each S-sphere formed with this value of h lies within the S-sphere formed with the value of $h=\frac{1}{2}\sqrt{3}$. It follows that C now touches no S-spheres throughout its entire length.

The invariant tube Γ. We shall now construct a sort of tube, Γ, enclosing C as follows: With P as center construct a circle, K, in a plane normal to C, and generate a surface by the motion of a circle through z_1, z_2, and a point which moves around the circumference of K. We shall take K small enough that the tube from the neighborhood of P to that of P' be exterior to S-spheres.

The tube is invariant under the transformation T. For, the arc through z_1 and z_2 which generates it is, in each position, an invariant arc.

The plane area enclosed by K is transformed by T into a spherical surface through P' bounded by a circle K'. If we apply repeatedly the transformation T and its inverse to the portion of the solid tube bounded by these two sections at P and P', its transforms fill up without overlapping the whole interior of Γ. It follows that any point within Γ is exterior to all S-spheres.

The tube terminates at z_1 in a conical point. Now take $\omega = z_1$, and construct the L-line there. This line, being normal to the z-plane, lies entirely within Γ in the neighborhood of z_1 and passes out of the tube at some point Q above z_1. Between z_1 and Q the line is exterior to all S-spheres. Above Q it can intersect only a finite number of S-spheres. Hence the inequality (2) is satisfied by only a finite number of rational fractions.

If we make a transformation of the type (3), z_1 is carried into an irrational z'. Γ is carried into a tube exterior to all S-spheres. The transformed tube has a conical point at z'; and the L-line erected at z' intersects only a finite number of S-spheres.

It follows that all points into which z_1 can be carried by transformations of the group of Picard are irrationals for which the inequality (2) is satisfied by only a finite number of rational fractions. But the transforms of this point, as of any point in the z-plane, are everywhere dense in the z-plane. This establishes the second part of the theorem.

THE RICE INSTITUTE,
HOUSTON, TEX.

SOLUTIONS OF THE EINSTEIN EQUATIONS INVOLVING FUNCTIONS OF ONLY ONE VARIABLE*

BY

EDWARD KASNER

Taking the element of length in the orthogonal form

$$ds^2 = \lambda_1\, dx_1^2 + \lambda_2\, dx_2^2 + \lambda_3\, dx_3^2 + \lambda_4\, dx_4^2 \tag{1}$$

we assume that the potentials λ are functions of a single variable x_1. We shall find the forms of these functions for which the Einstein cosmological equations

$$R_{ik} - \frac{1}{4} R g_{ik} = 0 \tag{2}$$

are satisfied. The results are given in (14), (20), and (17) below. They involve only elementary functions, and the corresponding four-dimensional manifolds can be represented as imbedded in flat space of seven dimensions.

Making use of the formulas for R_{ik} calculated in a previous paper† we find that, of the ten equations (2), the six with unlike subscript vanish identically, while the four with like subscripts become

$$\begin{aligned}
&L_{211} + L_{311} + L_{411} + L_{21}^2 + L_{31}^2 + L_{41}^2 - L_{21} L_{31} - L_{31} L_{41} \\
&\qquad - L_{41} L_{21} - L_{11}(L_{21} + L_{31} + L_{41}) = 0, \\
&L_{211} - L_{311} - L_{411} + L_{21}^2 - L_{31}^2 - L_{41}^2 + L_{21} L_{31} - L_{31} L_{41} \\
&\qquad + L_{41} L_{21} - L_{11}(L_{21} - L_{31} - L_{41}) = 0, \\
&-L_{211} + L_{311} - L_{411} - L_{21}^2 + L_{31}^2 - L_{41}^2 - L_{21} L_{31} + L_{31} L_{41} \\
&\qquad - L_{41} L_{21} - L_{11}(-L_{21} + L_{31} - L_{41}) = 0, \\
&-L_{211} + L_{311} + L_{411} - L_{21}^2 - L_{31}^2 + L_{41}^2 - L_{21} L_{31} - L_{31} L_{41} \\
&\qquad + L_{41} L_{21} - L_{11}(-L_{21} - L_{31} + L_{41}) = 0.
\end{aligned} \tag{3}$$

* Presented to the Society, September 7, 1921.

† Kasner, *The solar gravitational field completely determined by its light rays*, Mathematische Annalen, vol. 85 (1922), p. 227.

Here $L_i = \frac{1}{2}\log\lambda_i$ and the second and third subscripts indicate differentiation with respect to x_1, while the first subscript states merely the name of the function.

The four equations (3) are not independent, their sum being identically zero. They thus reduce to three equations. We may assume without loss of generality that $\lambda_1 = 1$, for by applying the transformation

$$\lambda_1\, dx_1^2 = dx_1^{*2}$$

to the form (1), it reduces to

$$ds^2 = dx_1^{*2} + \lambda_2^*\, dx_2^2 + \lambda_3^*\, dx_3^2 + \lambda_4^*\, dx_4^2, \tag{4}$$

where the λ_i^* are three new functions of the new variable x_1^*. Thus the functions L_i reduce to

$$L_1^* = 0, \qquad L_i^* = \frac{1}{2}\log\lambda_i^* \qquad (i = 2, 3, 4).$$

We then obtain from equations (3) by adding the first and second, first and third, and the first and fourth, respectively

$$\begin{aligned} L_{211}^* + L_{21}^{*2} - L_{31}^* L_{41}^* &= 0, \\ L_{311}^* + L_{31}^{*2} - L_{41}^* L_{21}^* &= 0, \\ L_{411}^* + L_{41}^{*2} - L_{21}^* L_{31}^* &= 0. \end{aligned} \tag{5}$$

We now introduce three new functions

$$x = L_{21}^*, \qquad y = L_{31}^*, \qquad z = L_{41}^*, \tag{5'}$$

and denote the independent variable x_1^* by t, thus reducing our system (5) to the simple form

$$\begin{aligned} x' &= yz - x^2, \\ y' &= zx - y^2, \\ z' &= xy - z^2, \end{aligned} \tag{6}$$

a system of three ordinary differential equations, primes denoting differentiation with respect to t.

SOLUTION OF THE SYSTEM (6)

In order to solve the system (6) we first reduce it by means of the substitution

$$(6') \qquad \xi = x + y + z, \qquad \eta = xy + yz + zx, \qquad \zeta = xyz$$

to the more easily integrable form

$$(7) \qquad \begin{aligned} \xi' + \xi^2 &= 3\eta, \\ \eta' &= 0, \\ \zeta' + 3\xi\zeta &= \eta^2. \end{aligned}$$

The second of these equations shows that η is constant and we write for later convenience

$$(8) \qquad \eta = -\frac{c^2}{3}.$$

Then the first becomes

$$(8') \qquad \xi' + \xi^2 + c^2 = 0,$$

the solution of which (unless ξ is constant, which special case is treated later)

$$(9) \qquad \xi = c \tan c\,(k - t),$$

where k is an arbitrary constant of integration. (The special case where $c = 0$, that is, $\eta = 0$, is treated separately later.)

By (6′) the three functions x, y, z are the roots of the cubic equation

$$(9') \qquad x^3 - \xi x^2 + \eta x - \zeta = 0.$$

Substituting

$$(9'') \qquad x = X + \frac{1}{3}\xi$$

and using (8) and (9), this reduces to

$$(10) \qquad X^3 - \frac{1}{3}c^2 \sec^2 c\,(k-t)\cdot X - \left(\zeta + \frac{2}{27}\xi^3 - \frac{1}{3}\xi\eta\right) = 0.$$

We observe that the third equation in (7) is linear in ζ and has for a particular solution*

$$(10') \qquad \zeta = -\frac{2}{27}\xi^3 + \frac{1}{3}\xi\eta.$$

* This was suggested to me by Dr. Gronwall, whom I wish to thank, as well as Dr. Reddick, for simplifications in my original calculations.

and hence we obtain its general solution in the form

$$\zeta = -\frac{2}{27}\xi^3 + \frac{1}{3}\xi\eta + c_1 c^3 \sec^3 c(k-t), \tag{11}$$

where c_1 is an arbitrary constant of integration. If now we divide the roots of the cubic (10) by $c\sec c(k-t)$, that is, employ the substitution

$$X = ac\sec c(k-t),$$

it becomes

$$a^3 - \frac{1}{3}a - c_1 = 0, \tag{12}$$

so that the new unknown a is actually constant and dependent on c_1. We then have as a general solution of equations (6)

$$x, y, z = \frac{1}{3}c\tan c(k-t) + a_i c\sec c(k-t), \tag{13}$$

where a_i takes on successively the values which are the roots of the cubic (12). The result thus involves three arbitrary constants c, k, c_1.

The principal quadratic form in one variable

Going back now to the original notation by means of (5′) and integrating the expressions for x, y, z given in (13), we obtain

$$L_i^* = \frac{1}{2}\log\lambda_i^* = \frac{1}{3}\log\cos(k-x_1^*) + a_i\log\{\sec c(k-x_1^*) + \tan c(k-x_1^*)\}.$$

Therefore

$$\lambda_i^* = \cos^{2/3} c(k-x_1^*)\{\sec c(k-x_1^*) + \tan c(k-x_1^*)\}^{2a_i} \qquad (i = 2, 3, 4)$$

are the general values of the coefficients in the differential form (4).

We now return to the general orthogonal form (1) by the transformation

$$dx_1^{*2} = \frac{4\,dx_1^2}{c^2(1+x_1^2)},$$

that is,

$$\cos c(k-x_1^*) = \frac{2x_1}{1+x_1^2}.$$

which eliminates all the trigonometric functions, giving

$$\lambda_1 = \frac{4}{c^2(1+x_1^2)^2}, \qquad \lambda_i = \left(\frac{2x_1}{1+x_1^2}\right)^{2/3} x_1^{2a_i} \qquad (i = 2, 3, 4).$$

We have thus established our main theorem:

The principal solution, in orthogonal form (1) *where the coefficients are functions of* x_1 *alone, of Einstein's cosmological equations*

$$R_{ik} - \frac{1}{4} g_{ik} R = 0, \tag{2}$$

is given by

$$ds^2 = \frac{4\,dx_1^2}{c^2(1+x_1^2)^2} + \left(\frac{2x_1}{1+x_1^2}\right)^{2/3} \{x_1^{2a_2}\,dx_2^2 + x_1^{2a_3}\,dx_3^2 + x_1^{2a_4}\,dx_4^2\}, \tag{14}$$

where c *is an arbitrary constant and* a_2, a_3, a_4 *are the roots of a cubic of type* (12), *that is, are constants obeying the relations*

$$\begin{aligned} a_2 + a_3 + a_4 &= 0, \\ a_2 a_3 + a_3 a_4 + a_4 a_2 &= -\frac{1}{3}. \end{aligned} \tag{14'}$$

The potentials in this form are in general intercendental (transcendental) functions, since the exponents are usually irrational. A simple example where the exponents are rational numbers and hence the potentials g_{ik} are algebraic functions is given by

$$a_2 = \frac{1}{3}, \qquad a_3 = \frac{1}{3}, \qquad a_4 = -\frac{2}{3}.$$

(This of course does not mean that the finite four-dimensional spread is necessarily algebraic.)

Special case where the constant c vanishes

In this special case $\eta = 0$, so that from (7) we find

$$\xi = \frac{1}{t - t_0} \quad \text{or} \quad \xi = 0.$$

For the first form we find $\zeta = c_2/(t-t_0)^3$, and finally,

$$\begin{gathered} x = \frac{\lambda}{t-t_0}, \qquad y = \frac{\mu}{t-t_0}, \qquad z = \frac{\nu}{t-t_0}, \\ \lambda + \mu + \nu = 1, \qquad \lambda\mu + \mu\nu + \nu\lambda = 0. \end{gathered} \tag{15}$$

For the second form, $\zeta = \text{constant} = c_3^3$, and

$$(16) \qquad x = c_3, \qquad y = \omega c_3, \qquad z = \omega^2 c_3,$$

where ω is an imaginary cube root of unity.

Thus the quadratic form is either

$$(15') \qquad ds^2 = dx_1^2 + x_1^{2\lambda} dx_2^2 + x_1^{2\mu} dx_3^2 + x_1^{2\nu} dx_4^2,$$

or else

$$(16') \qquad ds^2 = dx_1^2 + e^{2c_3x_1} dx_2^2 + e^{2\omega c_3 x_1} dx_3^2 + e^{2\omega^2 c_3 x_1} dx_4^2.$$

Both of these solutions may be included by transformation in the single form*

$$(17) \qquad \begin{gathered} ds^2 = x_1^{2a_1} dx_1^2 + x_1^{2a_2} dx_2^2 + x_1^{2a_3} dx_3^2 + x_1^{2a_4} dx_4^2, \\ a_2 + a_3 + a_4 = a_1 + 1, \qquad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2. \end{gathered}$$

The simplest solution in integers of the two conditions on the exponents in (17) is $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = -1$, giving

$$ds^2 = x_1^4 (dx_1^2 + dx_2^2 + dx_3^2) + x_1^{-2} dx_4^2.$$

By a misprint in the American Journal of Mathematics, the last term is incorrectly written $x_4^{-2} dx_4^2$.

SPECIAL CASE WHERE THE FUNCTION ξ IS CONSTANT

Going back to (8′), it was noted that the solution (9) does not include the case where ξ is constant. In that case, replacing ic by k, we have from (7)

$$(18) \qquad \xi = k, \qquad \eta = \frac{k^2}{3}, \qquad \zeta = \frac{k^3}{27} + C_1 e^{-3kt},$$

where C_1 is a new constant of integration. We then find

$$(19) \qquad \begin{aligned} x &= \frac{k}{3} + C_1 e^{-kt}, \\ y &= \frac{k}{3} + \omega C_1 e^{-kt}, \\ z &= \frac{k}{3} + \omega^2 C_1 e^{-kt}, \end{aligned}$$

* This result is the solution of the Einstein field equations in their first form, $R_{ik} = 0$, and was first given by the writer in a paper read before the Society in September, 1920 (see Bulletin of the American Mathematical Society, vol. 27 (1920–21), p. 62). See also the American Journal of Mathematics, vol. 43 (1921), p. 220. The same problem was later treated by Murnaghan and Eisenhart. The general results of the present paper were given in Science, vol. 54 (1921), pp. 304–305.

where ω is, as above, an imaginary cube root of unity. From these values, using (5′), we find the three functions L_i^*; and finally by a simple transformation and a change of constants the final form is obtained

$$(20)\qquad ds^2 = \frac{dx_1^2}{k^2 x_1^2} + x_1^{2/3}\left\{e^{Cx_1^{-1}}\, dx_2^2 + e^{\omega C x_1^{-1}}\, dx_3^2 + e^{\omega^2 C x_1^{-1}}\, dx_4^2\right\}.$$

Summary of solutions

For the symmetric system of differential equations (6) *all solutions are included in* (13), (19), (15) *and* (16). We observe that (13) involves three arbitrary constants, (19) and (15) involve two, and (16) involves only one.

For our original problem, all solutions of the cosmological equations are included in (14), (20), *and* (17).

We note that the results involve only elementary functions. In general the potentials, that is the coefficients λ_i in the expression ds^2, are transcendental, but for an infinitude of special values of the constants involved they reduce to algebraic functions.

A form which is equivalent to the main solution (14) is

$$ds^2 = dx_1^2 + \lambda_2\, dx_2^2 + \lambda_3\, dx_3^2 + \lambda_4\, dx_4^2,$$

$$\lambda_i = \left(\frac{1-T^2}{1+T^2}\right)\left(\frac{1-T}{1+T}\right)^{2\alpha_i}, \qquad T = \tan\frac{cx_1}{2},$$

where $\alpha_2, \alpha_3, \alpha_4$ are the roots of a cubic equation of the form

$$\alpha^3 - \frac{1}{3}\alpha - c_1 = 0,$$

the constant c_1 being arbitrary as well as c.

Representation in space of seven dimensions

We now prove the following result:

All the solutions obtained may be represented in finite form by manifolds of four dimensions imbedded in a flat space of seven dimensions, the finite equations involving integrals of elementary functions.

We first observe that all quadratic forms of the type

$$(21)\qquad ds^2 = \lambda_1(x_1)\, dx_1^2 + \lambda_2(x_1)\, dx_2^2 + \lambda_3(x_1)\, dx_3^2 + \lambda_1(x_1)\, dx_4^2$$

are of class three, that is, are reducible to the sum of seven squares of exact differentials.

We may write (21) as follows:

$$ds^2 = ds_2^2 + ds_3^2 + ds_4^2 = Q_2 + Q_3 + Q_4, \tag{21'}$$

where

$$ds_i^2 = Q_i = \frac{1}{3}\lambda_1(x_1)\,dx_1^2 + \lambda_i(x_1)\,dx_i^2 \qquad (i = 2, 3, 4).$$

Here Q_i is a quadratic form in only two variables, x_1 and x_i, and is the fundamental form of a surface of revolution. *Thus our element* (21) *is the sum of the elements of three surfaces of revolution.*

Since the coördinate x_1 is common to the three forms Q_i we may take the three surfaces about a common axis of revolution. Denote this by X_1, and construct the first surface in a three-flat $X_1\,X_2\,X_3$, the second surface in a three-flat $X_1\,X_4\,X_5$, the third surface in a three-flat $X_1\,X_6\,X_7$. Here $X_1, X_2, \cdots, X_7$ are cartesian coördinates in a flat space of seven dimensions. The three surfaces of rotation thus have a common axis but are contained in three distinct three-flats which are mutually orthogonal.

The general theory of what I have called separable or summable* quadratic differential forms in any number of variables is a subject of considerable interest. The problem is to represent the given form in n variables as the sum of two or more forms each involving fewer than n variables. We must distinguish the case where the variables used in the summands are completely independent, and the case where dependent variables are introduced. The problem discussed in the present paper comes under the latter case (since the three binary forms in (21′) do not involve completely independent variables), while the problem of an earlier paper in these Transactions† gives an example of complete separability.

* Kasner, *Separable quadratic differential forms and Einstein solutions*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 95–96, and abstracts in the Bulletin of the American Mathematical Society.

† *An algebraic solution of the Einstein equations*, vol. 27 (1925), pp. 101–105.

Columbia University,
New York, N. Y.

A GENERAL THEORY OF LINEAR SETS*

BY

MARK H. INGRAHAM

Introduction

Section I of the following paper, though using the postulational method, is motivated by the consideration of classes of vectors, on a finite range $P^n = (1, 2, \cdots, n)$, whose elements belong to a general division algebra or, as we shall say, number system.

Section II deals only with vectors on a finite range.

Section I is also of use as giving a general basis preliminary to the more intensive study of

(a) classes of vectors on a general range,

(b) number systems over a division number system; that is, to the initiation of a theory analogous to that of an "algebra over a field," where the field is replaced by an associative division number system.

Notation. Throughout the paper certain logical notations† will be used as follows:

$=$	logical identity
$\neq$	logical diversity
$\equiv$	definitional identity
$:\equiv:$	definitional identity between statements
$.).$	implies
$.\sim.$	is equivalent to
$.\ni.$	such that
$\exists$	there exists
$\vert$	is unique, used before the element which is unique: thus, $\vert a$ means a is unique.
$.$	and
$.\cup.$	or
$-$	not
. : .: :: etc.	punctuation signs; the principal implication of a sentence has its sign accompanied by the largest number of periods, thus $a :) : b .) . c$ is a statement that a implies that (b implies c) whereas $a .) . b :) : c$ states that the implication a implies b, implies the fact c. We may also use punctuation to show continued implication, thus $a .) . b .) . c$ means $a .) . b$ and $b .) . c$.

* Presented to the Society, April 11, 1925.

† These signs are mostly taken from E. H. Moore's *Introduction to a Form of General Analysis*. Yale University Press, 1910, p. 150.

$\sim$	corresponds to
$*(a)$	the statement $*$ holds for every a
$[\ ]$	class of elements. A non-vacuous class we call a set.
$\cap[P]$	the greatest common subclass of the classes P of the class $[P]$ of classes
$\cup[P]$	the least common superclass of the classes P of the class $[P]$ of classes
$\supset$	inclusion. In speaking of classes M and N, $M \supset N$ means M includes N, in the sense that every element of N is an element of M. This may also be written $N \subset M$.

The principal results will frequently be stated both in logical notation and in the written form; proofs, however, will as far as practical be given in logical notation only.

In dealing with subsets of the fundamental classes $\mathfrak{A}$, U, V, etc.,

$$\mathfrak{A}_0 = [a_0], \quad \mathfrak{A}_1 = [a_1], \quad \cdots, \quad U_0 = [u_0], \quad \cdots, \quad V_0 = [v_0], \quad \cdots \quad \text{etc.}$$

as subsets of $\mathfrak{A}$, U, V etc. respectively.

We shall use exponents to denote properties of an entity a; for example $\mathfrak{A}^A$ denotes that $\mathfrak{A}$ is of type A. When we use the notation for a class as an exponent of an element we shall mean that the element belongs to the given class; thus u_0^U means that u_0 is a member of the class U.

If we have a single valued function, f, of n independent variables whose values belong to the ranges $P_1, \cdots, P_n$, and the functional values of f belong to a class M then we say that the function f is on $P_1 \cdots P_n$ to M, that is $f^{\text{on } P_1 \cdots P_n \text{ to } M}$.

Number systems of type A. We will consider a number system which is a generalization of a "division algebra." We will define what we mean by a number system, $\mathfrak{A}$, being of type A, in such a way that whenever multiplication between every two elements of $\mathfrak{A}$ is commutative, it follows that $\mathfrak{A}$ is a field. If $\mathfrak{A}$ is a field we will say $\mathfrak{A}^F$. However as we do not assume that multiplication is commutative, there is introduced both a right and left distributive law.

We say that a system $\mathfrak{A}$ is a number system of *type* A or symbolically $\mathfrak{A}^A$ if it is of the following type:†

† This definition of a number system of type A is based directly on that used by E. H. Moore in his course in General Analysis. A number system of type A has properties 1–11 as given for real numbers by D. Hilbert but does not necessarily fulfill conditions 12–17. See D. Hilbert, *Grundlagen der Geometrie,* p. 35, 3d edition, 1909, or the translation by E. J. Townsend, *The Foundations of Geometry,* Open Court Publishing Company, 1902, p. 37.

$\mathfrak{A}$ contains at least two distinct elements;

There is an addition function, $+$, on $\mathfrak{A}\mathfrak{A}$ to $\mathfrak{A}$, which forms a commutative group, with identity of addition $= 0$;

There is a multiplication function, $\times$, on $\mathfrak{A}\mathfrak{A}$ to $\mathfrak{A}$ which obeys the following restrictions:

(1) $0 \times a = 0 = a \times 0 \quad (a)$;

(2) $\times$ on $\mathfrak{A}$ except 0 forms a group (not supposed commutative);

(3) $a_1 \times a_2 = 0 :) : a_1 = 0 .\cup. a_2 = 0$;

(4) The identity of multiplication $= 1$;

(5) $a_1 \times (a_2 + a_3) = (a_1 \times a_2) + (a_1 \times a_3) \quad (a_1, a_2, a_3)$.
$(a_2 + a_3) \times a_1 = (a_2 \times a_1) + (a_3 \times a_1) \quad (a_1, a_2, a_3)$.

We will call such a system an associative division number system.

For simplicity of notation we will write $a_1 \times a_2 = a_1 a_2$. We will use the exponential method of denoting reciprocals thus:

$$a_1 a_2 = a_2 a_1 = 1 : \; : a_2^{-1} = a_1 \,.\, a_1^{-1} = a_2 .$$

For sake of clarity a few examples of number systems of type A will be given.

Ex. 1. All real rational numbers.

Ex. 2. The system, R, of all real numbers.

Ex. 3. The system, C, of all complex numbers.

Ex. 4. The system, Q, of all real quaternions.

Ex. 5. Any Galois field $GF[p_n]$;
e. g. for $n = 1$ the rational numbers modulo p.

Ex. 6.* The Hilbert example of a non-Archimedian Veronesean number system.

Consider $P = (\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots)$ and a number system $\mathfrak{A}^A$.

Consider $F =$ [all single valued functions $f^{\text{on } P \text{ to } \mathfrak{A}} :: \supset :: f .:) :. \exists\, n_f : \supset : n < n_f .). f(n) = 0$].

For a definiton of addition we have

$$f = f_1 + f_2 : \; : f(n) = f_1(n) + f_2(n) \quad (n).$$

For a definition of multiplication we have

$$f = f_1 f_2 : \; : f(n) = \sum_{jk}^{j+k=n} f_1(j) f_2(k) \quad (n).$$

* This example is developed by E. H. Moore in his course in General Analysis. For $\mathfrak{A} = R$ this is Hilbert's example of a non-Archimedian Veronesean number system. Loc. cit., p. 31, or trans., p. 34.

The identity of addition is $[f_0(n) = 0 \quad (n)]$.

The identity of multiplication is $[f_1(0) = 1 \,.\, f_1(n) = 0 \quad (n \neq 0)]$.

F is of type A.

Ex. 7.* Consider $P = (\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots)$ and a number system $\mathfrak{A}$ of type A.

Consider $F[\text{all } f^{\text{on } PP \text{ to } \mathfrak{A}} \,.\!:\!:\!\ni\!:\!:.\, \exists\, n_f \,.\!:\!\ni\!:.\, n_1 \leqq n_f \,.\, n_2 \,.).\, f(n_1\, n_2) = 0 :: n$ $.:):.\, \exists\, n_{fn} :\ni: n_3 \leqq n_{fn} \,.).\, f(n\, n_3) = 0]$.

For a definition of addition we have

$$f = f_1 + f_2 \;\vdots\!=\!\vdots\; f(n_1, n_2) = f_1(n_1, n_2) + f_2(n_1, n_2) \quad (n_1, n_2).$$

For a definition of multiplication we have

$$f = f_1 f_2 \;\vdots\!=\!\vdots\; f(n_1\, n_2) = \sum_{k_1 k_2 m_1 m_2}^{\substack{k_1 + m_1 = n_1 \\ k_2 + m_2 = n_2}} a^{k_1 m_2} f_1(k_1, k_2)\, f_2(m_1, m_2) \quad (n_1, n_2),$$

where a is a number of $\mathfrak{A}$.

Our identity of addition is $[f_0(n_1, n_2) = 0 \;(n_1, n_2)]$.

Our identity of multiplication is $[f_1(0, 0) = 1 \,.\, f_1(n_1, n_2) = 0(n_1 \neq 0$ $.\cup.\, n_2 \neq 0)]$.

F is of type A.†

Theorem 1. *If $\mathfrak{A}$ is of type A, and $\mathfrak{A}_0$ is a subset of A, then the totality $\mathfrak{A}_{0c}$, of numbers of $\mathfrak{A}$ which are commutative as to multiplication with all the numbers of $\mathfrak{A}_0$, forms with the original addition and multiplication a number system of type A*:

$$\text{Th. 1.} \quad \mathfrak{A}^A \,.\, \mathfrak{A}_0 = [a_0] \,.\, \mathfrak{A}_{0c} = [\text{all } a :\ni: a_0 \,.).\, a\, a_0 = a_0\, a] :): \mathfrak{A}_{0c}^A.$$

The proof is evident when we note that $a^{\mathfrak{A}_{0c}} \,.).\, (a^{-1})^{\mathfrak{A}_{0c}}$ for $a\, a_0 = a_0\, a \,.).\, a_0 = a^{-1} a_0\, a \,.).\, a_0\, a^{-1} = a^{-1} a_0 \;(a_0)$.

Corollary. *If $\mathfrak{A}$ is of type A, then the totality $\mathfrak{A}'$ of numbers of $\mathfrak{A}$ which are commutative as to multiplication with all the numbers of $\mathfrak{A}$ forms, with the original addition and multiplication, a field*:

$$\text{Cor.} \qquad \mathfrak{A}^A \,.\, \mathfrak{A}' = [\text{all } a :\ni: a_1 \,.).\, a\, a_1 = a_1\, a] :): \mathfrak{A}'^F.$$

---

* A special case of this example is given by Hilbert, loc. cit., p. 100, or trans., p. 103. J. H. M. Wedderburn brought this example to my attention in a more general form than that of Hilbert.

† The proof of this directly follows Hilbert's proof. It should be noted that in both Ex. 6 and Ex. 7 Hilbert uses formal power series in one or two symbolic parameters respectively, rather than the functional notation used here.

It should be noted that $\mathfrak{A}'$ is not necessarily a maximal field in $\mathfrak{A}$. If, for example, $\mathfrak{A} = Q$ then $\mathfrak{A}' =$ scalars but all the quaternions such that the coefficients of j and k are 0 form a field isomorphic with C and containing the scalars.

$\mathfrak{A}$ may be of infinite rank in respect to $\mathfrak{A}'$ as is shown by Example 7. Thus a number system of finite rank as regards a number system $\mathfrak{A}$ might be of infinite rank as regards the field $\mathfrak{A}'$ of which $\mathfrak{A}$ is an extension.

I. The theory of linear sets

Contents

1. **General postulational basis.** In Section 1 we consider a system

$$\Sigma = (\mathfrak{A}, U = [u], \oplus, \odot_r, \odot_l),$$

viz., a number system $\mathfrak{A}$ of type A, a class U of (abstract vectors or) elements u and three processes or functions $\oplus$, $\odot_r$, $\odot_l$, serving to connect numbers a and elements, u; as follows:

1. U has at least two distinct elements.
2. $\oplus$ is a function on UU to U which forms a commutative group with identity element 0_U; $u = u_1 \oplus u_2$, u is the sum of u_1 and u_2.
3. $\odot_r$ is a function on $U\mathfrak{A}$ to U; $u = u_1 \odot_r a$; u is the product of u_1 by a (on the right).
4. $\odot_l$ is a function on $\mathfrak{A}U$ to U; $u = a \odot_l u_1$; u is the product by a (on the left) of u_1.
5. $a\,.).\, a \odot_l 0_U = 0_U = 0_U \odot_r a$.
6. $u\,.).\, 0 \odot_l u = 0_U = u \odot_r 0$.
7. $u\,.).\, u \odot_r 1 = u = 1 \odot_l u$.
8. Associative law of addition.

$$u_1\, u_2\, u_3\,.).\, (u_1 \oplus u_2) \oplus u_3 = u_1 \oplus (u_2 \oplus u_3).$$

9. Distributive law,

$$\begin{aligned} u_1\, u_2\, a_1\, a_2\,.).\, &(a)\ (u_1 \odot_r a_1) \oplus (u_1 \odot_r a_2) = u_1 \odot_r (a_1 + a_2),\\ &(b)\ (a_1 \odot_l u_1) \oplus (a_2 \odot_l u_1) = (a_1 + a_2) \odot_l u_1,\\ &(c)\ (u_1 \odot_r a_1) \oplus (u_2 \odot_r a_1) = (u_1 \oplus u_2) \odot_r a_1,\\ &(d)\ (a_1 \odot_l u_1) \oplus (a_1 \odot_l u_2) = a_1 \odot_l (u_1 \oplus u_2).\end{aligned}$$

10. Associative law of multiplication,

$$u\,a_1\,a_2\,.).\ \text{(a)}\ (u \odot_r a_1) \odot_r a_2 = u \odot_r (a_1\, a_2),$$
$$\text{(b)}\ a_1 \odot_l (a_2 \odot_l u) = (a_1\, a_2) \odot_l u,$$
$$\text{(c)}\ (a_1 \odot_l u) \odot_r a_2 = a_1 \odot_l (u \odot_r a_2).$$

From 9 and 10 the general distributive and associative laws follow. As it will not lead to ambiguity we will simplify the notations as follows:

$$\begin{array}{llll} u_1 \oplus u_2 = u_1 + u_2 & & & (u_1\, u_2); \\ u \odot_r a = u a & & & (u\, a); \\ a \odot_l u = a u & & & (u\, a); \\ (u a_1)\, a_2 = u\,(a_1\, a_2) & = u a_1\, a_2 & & (u\, a_1\, a_2); \\ a_1\,(a_2\, u) = (a_1\, a_2)\, u & = a_1\, a_2\, u & & (u\, a_1\, a_2); \\ a_1\,(u a_2) = (a_1\, u)\, a_2 & = a_1\, u a_2 & & (u\, a_1\, a_2). \end{array}$$

There follow two examples of a system Σ.

Ex. 1.* Any algebra over a field.

It should be noted that our system Σ is more general in that we have not limited $\mathfrak{A}$ to be a field and we have not required the existence of a multiplication process between the elements of U ($\odot$ on UU to U).

Ex. 2. If we are given a general range P and a number system $\mathfrak{A}$ of type A, then we may take as U the set F of all vectors f (single valued functions), on P to $\mathfrak{A}$,—

$$\begin{array}{lll} F = [\text{all } f^{\text{on } P \text{ to } \mathfrak{A}}], \ \text{with} & & \\ f = f_1 + f_2 \ :=: f(p) = f_1(p) + f_2(p) & & (p), \\ f = a f_1 \ :=: f(p) = a f_1(p) & & (p), \\ f = f_1 a \ :=: f(p) = f_1(p)\, a & & (p). \end{array}$$

The symmetry between $\odot_r$ and $\odot_l$, and the symmetry between right and left multiplication in $\mathfrak{A}$ should be noted. Each theorem will carry with it a theorem by parity (not always different). As a convention, theorems involving only one type of multiplication will be stated in terms of right hand multiplication.

From Postulate 2 we know that

$$u:):\exists\,|u_1\ .\ni.\ u + u_1 = u_1 + u = 0_U;$$

* See L. E. Dickson, *Algebras and their Arithmetics*, University of Chicago Press, 1923, p. 9.

this uniquely existing u_1 we designate the *negative* of u, in notation, $-u$, so that

$$u+u_1 = 0_U \;:=:\; u_1 = -u.$$

THEOREM 1. *The negative of any element of U is that element multiplied on either right or left by the number -1:*

TH. 1. $$u\,.).\,(-1)\,u = -u = u\,(-1).$$

Proof. $u = 1u$; then by Postulates 7, 8 and 9,

$$u+(-1)u = 1u+(-1)u = (1-1)u = 0u = 0_U.$$

THEOREM 2. *If the product of an element u of U by a number a of $\mathfrak{A}$ is 0_U then either a is 0 or u is 0_U or both:*

TH. 2. $$au = 0_U :): a = 0 \,.\cup.\, u = 0_U.$$

Proof. $a \neq 0 :): au = 0_U\,.).\, u = a^{-1}\,au = 0_U.$

THEOREM 3. *Relative to a subset U_0 of U, the totality $\mathfrak{A}_0$ of numbers a of $\mathfrak{A}$ which are commutative as to multiplication with all the elements of U_0 forms, with the original addition and multiplication, a number system A_0 of type A:*

TH. 3. $$U_0 :): (\mathfrak{A}_0 = [\text{all } a :\ni: u_0\,.).\, au_0 = u_0 a])^A.$$

Proof.

(1) $a_1 u = ua_1 \,.\, a_2 u = ua_2 :): (a_1+a_2)u = u(a_1+a_2)\,.\,a_1 a_2 u = ua_1 a_2,$

for

$$(a_1+a_2)u = a_1 u + a_2 u = ua_1 + ua_2 = u(a_1+a_2),$$

$$(a_1 a_2)u = a_1(a_2 u) = a_1(ua_2) = (a_1 u)a_2 = (ua_1)a_2 = u(a_1 a_2).$$

(2) $$au = ua\,.).\, a^{-1}u = ua^{-1}$$

for

$$u = a^{-1}\,ua \text{ and therefore } a^{-1}u = ua^{-1}.$$

(3) The set $\mathfrak{A}_0$ contains at least two numbers, for it contains 0 and 1.

Other properties of $\mathfrak{A}^A$ may be readily checked.

Definition. A set U_0 is said to be commutative, U_0^c, in case every number a is commutative with every element u_0 of U_0:

Def. $$U_0^c :=: au_0\,.).\, au_0 = u_0 a.$$

2. Sum and intersection of two sets, supplementary sets, additive sets, linear sets.

Definition of the *sum* **of two sets** U_1, U_2.

$$U_1 + U_2 = [\text{all } u :\ni: \exists\, u_1 u_2 .\ni. u = u_1 + u_2].$$

Definition of the *intersection* **of two sets** U_1, U_2. The intersection of $U_1 U_2$, $\cap [U_1 U_2]$, is the greatest common subset of U_1 and U_2.

Definition of *supplementary sets*. Two sets U_1 and U_2 are supplementary, $(U_1 U_2)^{\text{sup}}$, if their sum is U and their intersection is the set whose only element is 0_U.

Definitions* of *right linear* (rl), *left linear* (ll),† *properly linear* (l), **and** *additive*‡ (ad) **sets.**

$$U_0^{rl} :=: a_1\, a_2\, u_{01}\, u_{02} .).\ (u_{01} a_1 + u_{02} a_2) \text{ belongs to } U_0;$$

$$U_0^{ll} :=: a_1\, a_2\, u_{01}\, u_{02} .).\ (a_1 u_{01} + a_2 u_{02}) \text{ belongs to } U_0;$$

$$U_0^{l} :=: U_0^{rl.\,ll};$$

$$U_0^{ad} :=: u_{01}\, u_{02} .).\ (u_{01} + u_{02}) \text{ belongs to } U_0 \text{ and } u_0 .).\ (-u_0) \text{ belongs to } U_0.$$

It should be noted that any properly linear subset U_0 of U, other than the set consisting of the single element 0_U, together with the number system $\mathfrak{A}$ and the original definition of addition and multiplication forms a system Σ satisfying the postulates of § 1. In the sequel we shall often make use of this fact by applying theorems stated in terms of U to a properly linear subset U_0 of U.

* We have not included in the text a definition of a linear set. A satisfactory definition of linearity would be such that any right (left) linear set is a linear set, and in case U is commutative should reduce to the definition of right (left) linearity. In arriving at such a definition we make use of the number system $\mathfrak{A}_c$ consisting of all numbers a of $\mathfrak{A}$ which are commutative with every element u of U. $\mathfrak{A}_c$ is a field contained in $\mathfrak{A}'$, for by § 1, Theorem 3, $\mathfrak{A}_c$ is of type $\mathfrak{A}$, and

$$a_c\, a\, u .).\ (a_c a) u = a_c (a u) = (a u) a_c = a (u a_c) = a (a_c u) = (a a_c) u .).\ a_c a = a a_c.$$

We will say that a subset U_0 of U is linear in case for every pair of numbers a_{c1} and a_{c2} in $\mathfrak{A}_c$ and every pair of elements u_{01} and u_{02} of U_0 the element $u_{01} a_{c1} + u_{02} a_{c2}$ belongs to U_0.

† Note the symmetry of right and left linearity. When a theorem concerns only one of the two we will state it in terms of right linearity and omit the parity theorem in terms of left linearity.

‡ This is a strong form of the definition. One might use the first condition alone as a weaker form.

THEOREM 1. *U is properly linear.*

THEOREM 2. *A right (or left) linear set is additive.*

THEOREM 3. *Every additive set contains 0_U; hence, the intersection of two additive sets is non-vacuous, and each of two additive sets is contained in their sum.*

THEOREM 4. *If two sets U_1 and U_2 are additive (right linear, left linear or properly linear) then their sum and intersection are additive (right linear, left linear or properly linear).*

THEOREM 5. *If U_1 and U_2 are additive sets and their intersection is the set whose only element is 0_U, then any element in their sum can be expressed in one and only one way as the sum of one element of U_1 and one element of U_2:*

TH. 5. $$U_1^{ad} \,.\, U_2^{ad} \,.\, \cap \,|\, U_1\, U_2 \,| = \text{the class } 0_U$$
$$.:):.\ u_{11}^{U_1} \,.\, u_{12}^{U_1} \,.\, u_{21}^{U_2} \,.\, u_{22}^{U_2} \,.\, u_{11} + u_{21} = u_{12} + u_{22}$$
$$:):\ u_{11} = u_{12} \,.\, u_{21} = u_{22}\,.$$

Proof. $u_{11} - u_{12} = u_{22} - u_{21}$. Then, since U_1 and U_2 are additive, $u_{11} - u_{12}$ belongs to U_1 and $u_{22} - u_{21}$ belongs to U_2, hence $u_{11} - u_{12}$ belongs to $\cap \,|\, U_1\, U_2 \,|$. Therefore $u_{11} - u_{12} = 0_U$ and $u_{11} = u_{12}$ and hence $u_{21} = u_{22}$.

THEOREM 6. *Relative to a subset $\mathfrak{A}_0$ of $\mathfrak{A}$, the class U_0 of all elements u_0 of U which are commutative with every number a_0 of $\mathfrak{A}_0$ is additive and is properly linear in respect to the set $\mathfrak{A}_{0c}$ of all numbers of $\mathfrak{A}$ which are commutative as to multiplication with every number of $\mathfrak{A}_0$.*

Proof. From the distributive and associative laws it follows that

$$u_1\, a_0 = a_0\, u_1 \,.\, u_2\, a_0 = a_0\, u_2 \,.\, a_0\, a_1 = a_1\, a_0 \,.\, a_0\, a_2 = a_2\, a_0$$
$$:):\ (u_1\, a_1 + u_2\, a_2)\, a_0 = a_0 (u_1\, a_1 + u_2\, a_2)\,.$$

Hence, since $\mathfrak{A}_{0c}$ contains the numbers 1 and -1, U_0 is additive and is properly linear in respect to $\mathfrak{A}_{0c}$.

In his *Introduction to a Form of General Analysis*,* E. H. Moore introduces the notion of the extensional attainability of a property P, defined for the subclasses M_0 of M. Considering a class M and a property P defined for the subclasses M_0 of M, we say that the property P is *extensionally attainable* in case "for every subclass M_0 of M there exists a class M_{0P} containing M_0 and contained in M, having the property P and such that every subclass of M which contains M_0 and has the property P contains M_{0P},"

* Loc. cit., p. 54.

or, what is equivalent, (1) M has the property P, and (2) for every class M_0 the greatest common subclass of all classes containing M_0 and having the property P has the property P. For such a property P, the *P-extension of* M_0 is the class M_{0P} of the first definition and also the greatest common subclass etc., of the second definition: In notation

$$M_{0P} = \cap \,[\text{all } M_1^P \,.\ni.\, M_1 \supset M_0].$$

It is important to note that $M_{0P} \supset M_1 \supset M_0$:): $M_{0P} = M_{1P}$.

THEOREM 7. *The properties of additivity, right linearity, left linearity and proper linearity are extensionally attainable in* U.

Proof. (The proof is given for right linearity only.)

(1) U is right linear by Theorem 1.

(2) $U_0 \,.\, U_{0r} = \cap\,[\text{all } U_1^{rl} \,.\ni.\, U_1 \supset U_0]$:): U_{0r}^{rl} by definition of right linearity.

Accordingly we introduce notations, for the various extensions of U_0 as follows:

AdU_0 = the additive extension of U_0;
L_rU_0 = the right linear extension of U_0;
$L_l\,U_0$ = the left linear extension of U_0;
LU_0 = the properly linear extension of U_0.

THEOREM 8. *The right linear extension of any subset* U_0 *of* U *is the totality of all right linear combinations of elements of* U_0:

$$\text{TH. 8.}\quad U_0 \,.\, U_{0r} = \left[\text{all } u :\ni: \exists\; n_u\, a_1, \cdots, a_{n_u}\, u_{01}, \cdots, u_{0n_u} \,.\ni.\, u = \sum_i^{1, n_u} u_{0i}\, a_i\right]$$
$$:):\; U_{0r} = L_r\, U_0.$$

Proof. Obviously $L_r\,U_0 \supset U_{0r}$, and $U_{0r} \supset U_0$ and is right linear, and therefore $U_0 \supset L_r\,U_0$.

Due to the symmetry between right and left linearity we will in general state theorems involving only L_r or L_l in terms of L_r and omit the theorem that follows by parity.

3. **Interrelation of certain extensions of U_0.** In this section we consider the iteration of the four processes Ad; L_r; L_l; L and a new process L_0; where $L_0\,U_0$ is defined as the (provably existent) maximal properly linear subset of the intersection of the right and left linear extensions of U_0. Moreover, it is shown that these processes along with the iterations $L_0\,L_r$ and $L_0\,L_l$ are closed under further iteration.

We also consider how far the seven sets AdU_0, $L_r U_0$ etc. are determined from a knowledge of certain of them.

THEOREM 1. *The properly linear extension of any subset U_0 of U is the right linear extension of the left linear extension of U_0 and by parity the left linear extension of the right linear extension of U_0:*

$$\text{TH. 1.} \qquad U_0 .). \; L_r L_l U_0 = LU_0 = L_l L_r U_0.$$

Proof. (1) $LU_0 \supset L_r L_l U_0$, for $LU_0 \supset L_l U_0 \supset U_0$ and therefore $LU_0 = LL_l U_0 \supset L_r L_l U_0$.

(2) $L_r L_l U_0$ is properly linear, for by definition $U_1 .). (L_r U_1)^{rl}$ and $L_r L_l U_0$ is left linear, for according to § 2, Theorem 8, every element of $L_r L_l U_0$ is of the form

$$\sum_i^{1,n_0}\left(\sum_j^{1,n_i} a_{ji}\, u_{0ji}\right) a_i,^*$$

where n_0 and n_i (i) are positive integers and u_{0ji} belongs to U_0 $(i\,j)$ and conversely, and the distributive and associative laws are holding.

Therefore $L_r L_l U_0 = LU_0$ and similarly $L_l L_r U_0 = LU_0$.

THEOREM 2. $U_1 \, U_2$:): (1) $L_r \mathbf{U} [U_1 \, U_2] = L_r U_1 + L_r U_2$;
(2) $L\mathbf{U} [U_1 \, U_2] = LU_1 + LU_2$;
(3) $Ad\mathbf{U} [U_1 \, U_2] = AdU_1 + AdU_2$.

The proof follows directly from § 2, Theorem 8 and Theorem 1.

THEOREM 3. $U_1^{ad} \, U_2^{ad}$:): (1) $L_r [U_1 + U_2] = L_r U_1 + L_r U_2$;
(2) $L[U_1 + U_2] = LU_1 + LU_2$;
(3) $Ad[U_1 + U_2] = AdU_1 + AdU_2$.

THEOREM 4. *Relative to an additive subset U_0 of U, there exists a unique maximal properly linear subset U_{00} of U_0 such that all properly linear subsets of U_0 are contained in U_{00}:*

$$\text{TH. 4.} \qquad U_0^{ad} .:):. \exists \, | U_{00}^{l} \subset U_0 :\ni: U_{01}^{l} \subset U_0 .). \; U_{00} \supset U_{01}.$$

Proof. $U_1 = [\text{all } u .\ni. Lu \subset U_0]$ is effective as U_{00}; for
(1) U_1 contains the properly linear class $[0_U]$;
(2) $U_{01}^{l} \subset U_0 .). \; U_1 \supset U_{01}$;

---

* This may also be written in the form

$$\sum_i^{1,n} a_{1i}\, u_{0i}\, a_{2i}.$$

In this case we may not assume as above that the elements u_{0i} are distinct.

(3) U_1 is properly linear, for a right or left linear combination of elements of U_1 is the sum of elements of U_1 and therefore $LU_1 = AdU_1$ and $U_0 \supset LU_1$, hence by (2) $U_1 \supset LU_1$ and is properly linear.

Since relative to a subset U_0 of U, $\cap\,[L_r U_0\; L_l U_0]$ is additive, it follows from Theorem 4 that the following definition has content:

Definition of $L_0 U_0$. Relative to a subset U_0 of U we define $L_0 U_0$ as the maximal properly linear subset contained in the intersection of the right and left linear extensions of U_0.

THEOREM 5. *Relative to a right linear subset U_0 of U the maximal properly linear subset U_{00} of U_0 is $L_0 U_0$.*

Proof. U_0^{rl} .). $L_l U_0 \supset L_r U_0 = U_0$.

THEOREM 6. U_0 .). $L_0 U_0 = \cap\,[L_0 L_r U_0, L_0 L_l U_0]$.

The following table shows the sets generated from a set U_0 by iterated processes of the types Ad, L_r, L_l, L_0. For example, column 3, row 6 shows us that U_0 .). $L_0 L_r (L_l U_0) = L U_0$.

TABLE I

	Ad	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
Ad	Ad	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
L_r	L_r	L_r	L	L	L_0	$L_0 L_r$	$L_0 L_l$
L_l	L_l	L	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
L	L	L	L	L	L_0	$L_0 L_r$	$L_0 L_l$
L_0	L_0	$L_0 L_r$	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$
$L_0 L_r$	$L_0 L_r$	$L_0 L_r$	L	L	L_0	$L_0 L_r$	$L_0 L_l$
$L_0 L_l$	$L_0 L_l$	L	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$

The proof of the above table is readily obtained when we bear in mind that U_0 .). $\cap\,[L_r U_0\; L_l U_0] \supset Ad U_0 \supset U_0$.

The fact that relative to a given U_0 the seven sets $Ad U_0$ etc. may be distinct is shown by an example following Table I in II, § 2.

From an examination of Table I it is seen that the iteration of the seven processes Ad, etc., is closed and associative. Moreover, the iteration of the four processes Ad, L_r, L_l, and L is closed.

Table II shows which of the seven sets $Ad U_0$ etc., previously introduced, are determined when any particular combination of them is given. We do not list all the $2^7 - 1$ different combinations, but we give certain combinations, into which all of the $2^7 - 1$ can be decomposed, and such that no combination will determine, in general, more than could be determined from the component parts as listed in the table.

TABLE II

Combinations of sets given	Sets determined uniquely
Ad	$Ad, L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$
L_r	$L_r, L, L_0 L_r$
L_l	$L_l, L, L_0 L_l$
L	L
L_0	L_0
$L_0 L_r$	$L_0 L_r$
$L_0 L_l$	$L_0 L_l$
L_r, L_l	$L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$
$L_r, L_0 L_l$	$L_r, L_0 L_l, L, L_0 L_r, L_0$
$L_l, L_0 L_r$	$L_l, L_0 L_r, L, L_0 L_l, L_0$
$L_0 L_r, L_0 L_l$	$L_0 L_r, L_0 L_l, L_0$

The proof that the table is correct follows at once from Table I and Theorem 6.

The following considerations and examples show that Table II is complete: Whenever the determination of certain sets determines certain others uniquely it follows that no extra knowledge is gained by adding these others to the original sets in the first column of Table II. Thus, since the determination of $L_r U_0$ determines $L_0 L_r U_0$ it follows that we need not add L_r, $L_0 L_r$ to the combinations in the first column.

In Table III we give examples showing the completeness of Table II. In each of these examples we consider a finite range P^n, where n is a positive integer, $\mathfrak{A} = Q$ (quaternions) and $U = V$ the class of all vectors v on P to Q. We have a system satisfying the postulates of § 1 when multiplication and addition are defined as in § 1, Ex. 2 of a system Σ. We will display the vectors v as rows of ordered elements thus: $(a_i (i = 1, \cdots, 5)) = (a_1\, a_2\, a_3\, a_4\, a_5)$.

In these examples we display subsets V_1 and V_2 of such a nature that the sets arising from V_1 by processes listed in column 2 of Table III are equal respectively to those arising from V_2 by the same processes; but the sets arising from V_1 differ respectively from those arising from V_2 except for sets as shown by Table II to be uniquely determinable from those we have assumed to be equal. Thus, in Example 1, $L_r V_1$, LV_1, $L_0 L_r V_1$ are respectively equal to $L_r V_2$, LV_2, $L_0 L_r V_2$, but AdV_1, $L_l V_1$, $L_0 V_1$, $L_0 L_l V_1$ differ respectively from AdV_2, $L_l V_2$, $L_0 V_2$, $L_0 L_l V_2$.

When an example in Table III is given showing the completeness of Table II for any particular combination of sets AdV_0 etc. the example for

the same sets with left and right interchanged is immediately securable by parity.

TABLE III

Ex. No.	Example shows completeness of Table II for	P^n n	V_1	V_2
1.	$L_r\ (L_l)$	2	$(1\ \ 0)\ \ (0\ \ 1)$	$(1\ \ i)\ \ (j\ \ -k)$
2.	L	2	$(1\ \ 0)\ \ (0\ \ 1)$	$(1\ \ i)$
3.	L_0	3	$(0\ \ 0\ \ 0)$	$(1\ \ i\ \ j)\ \ (k\ \ -j\ \ -i)$
4.	$L_0\, L_r\ \ \ (L_0\, L_l)$	5	$(1\ \ i\ \ 0\ \ 0\ \ j)$ $(0\ \ 0\ \ 1\ \ 0\ \ 0)$ $(0\ \ 0\ \ 0\ \ 1\ \ 0)$	$(1\ \ j\ \ 0\ \ 0\ \ 0)$ $(0\ \ 0\ \ 1\ \ i\ \ 0)$ $(0\ \ 0\ \ j\ \ -k\ \ 0)$
5.	$L_r,\ \ L_l$	1	(1)	(i)
6.	$L_r,\ \ L_0\ \ (L_l,\ \ L_0)$	2	$(1\ \ i)\ \ (j\ \ k)$	$(1\ \ i)$
7.	$L_r,\ \ L_0\, L_l$ $(L_l,\ \ L_0\, L_r)$	2	$(1\ \ i)\ \ (j\ \ -k)$	$(1\ \ j)\ \ (i\ \ k)$
8.	$L,\ \ L_0$	2	$(1\ \ i)\ \ (j\ \ k)$	$(1\ \ j)\ \ (i\ \ k)$
9.	$L,\ \ L_0\, L_r$ $(L,\ \ L_0\, L_l)$	4	$(1\ \ 0\ \ 0\ \ 0)$ $(0\ \ 1\ \ 0\ \ 0)$ $(0\ \ 0\ \ 1\ \ i)$	$(1\ \ i\ \ 0\ \ 0)$ $(j\ \ -k\ \ 0\ \ 0)$ $(0\ \ 0\ \ 1\ \ j)$
10.	$L_0,\ \ L_0\, L_r$ $(L_0,\ \ L_0\, L_l)$	3	$(1\ \ i\ \ 0)\ \ (j\ \ k\ \ 0)$	$(1\ \ 0\ \ i)$
11.	$L_0\, L_r,\ \ L_0\, L_l$	3	$(1\ \ i\ \ i)$	$(1\ \ i\ \ j)$
12.	$L,\ \ L_0,\ \ L_0\, L_r$ $(L,\ \ L_0,\ \ L_0\, L_l)$	2	$(1\ \ i)\ \ (j\ \ k)$	$(1\ \ j)$
13.	$L,\ \ L_0\, L_r,\ \ L_0\, L_l$	2	$(1\ \ i)$	$(1\ \ j)$

4. **Right (left) linear independence, normal order, basis, rank, supplementary and difference sets.** In this section we consider a system Σ such that the set U is normally ordered. In this case we show that a right linear subset U_0 of U contains a right linearly independent base, and that every such base has the same cardinal number, which we call the right rank of U_0. Moreover we show that relative to a right linear subset U_0 of U there exists a supplementary right linear set; and that relative to a properly linear subset U_0 of U all properly linear supplementary sets are isomorphic with the difference set $U-U_0$ which in definition is analogous to a difference algebra.

We say a subset U_0 of U is *right linearly independent** as to $\mathfrak{A}_0$ in case

$$U_0\, \mathfrak{A}_0 \;.:\ni:.\; n\, (u_{01} \cdots u_{0n})^{\text{distinct}}\, a_{01} \cdots a_{0n}$$
$$:): \sum_i^{1,n} u_{0i}\, a_{0i} = 0_U \;.).\; a_{0i} = 0 \qquad (i = 1, \cdots, n).$$

In case U_0 is right linearly independent as to $\mathfrak{A}$ we say U_0 is right linearly independent (U_0^{rli}). If U_0 is right linearly dependent we use the notation U_0^{rld}. We say U_0 is right linearly independent as to U_1 in case $L_r\, U_1$ does not contain any members of U_0. Definitions of *left linear independence* as to $\mathfrak{A}_0$ etc. follow at once by parity.

THEOREM 1. *If a subset U_0 of U is commutative and right linearly independent as to $\mathfrak{A}'$, U_0 is right linearly independent.*

Proof. 1. U_0 does not contain 0_U.

2. If the theorem were not true we would have $\sum_i u_{0i}\, a_i = 0_U$, where the elements u_{0i} are distinct and the numbers a_i are $\neq 0$. Then

$$\sum_i u_{0i}\, a_i\, a_n^{-1} + u_{0n} = 0_U; \tag{1}$$

moreover since $u_{01}, \cdots, u_{0n}$ are right linearly independent as to $\mathfrak{A}'$ $\exists\, (j\ a \neq 0) \;.\ni.\; (a a_j\, a_n^{-1}\, a^{-1} - a_j\, a_n^{-1}) \neq 0$ and since U_0 is commutative

$$\sum_i u_{0i}\, a a_i\, a_n^{-1}\, a^{-1} + u_{0n} = 0_U; \tag{2}$$

hence from (1) and (2) it follows that

$$\sum_i u_{0i}\, (a a_i\, a_n^{-1}\, a^{-1} - a_i\, a_n^{-1}) = 0_U,$$

and therefore

$$\exists\; n_1 :\ni:\; n > n_1 > 0 \qquad (u_{01}, \cdots u_{0n_1})^{\text{distinct}\,.\,rld}.$$

By repetition of the above reasoning we conclude that

$$\exists\, (u_0 \neq 0_U\ a \neq 0) \;.\ni.\; u_0\, a = 0_U,$$

and since this is impossible our theorem is valid.

In the remainder of this section we will make use of the notion of normal order. "Normally ordered" is here used synonymously with well ordered

* This could more exactly be called finite right linear independence but as we do not consider the infinite case in this paper we shall use the shorter term.

(wohlgeordnet). Thus we say a subset U_0 of U is normally ordered (U_0^{no}) in case U_0 is linearly ordered in such a way that every subset U_{01} has a first element. We shall use the notation U_{0u}, where u is an element of a normally ordered set containing U_0, to denote all the elements of U_0 which precede u.

We recognize that many accept the Zermelo principle of selection or the multiplicative axiom and therefore feel that the Zermelo demonstration of the normal orderability of any aggregate warrants the assumption of such normal orderability without further stipulation; yet as many do not agree with this point of view we will introduce normal order explicitly whenever we wish to make use of it.

Definition. We say that U_1 is a *right base* for U_0 in case $L_r\,U_1 = U_0$. If, besides, U_1 is right linearly independent we say that U_1 is a *proper right* base for U_0. The definitions of a *left base* and a *proper left base* follow at once by parity.

THEOREM 2. *If U is normally ordered and if U_* is the set of all elements u of U which are right linearly independent of the preceding elements then U_* is a proper right base for U.*

Proof. 1. U_* is non-vacuous, for U has a first element different from 0_U.

2. $L_r\,U_* = U$. We use the indirect method of proof for this. If it is not so, then $\exists\, u \,.\ni.\, L_r\,U_*$ does not contain u, and hence there exists a first such u, say u_0. Since u_0 is not a member of U_* it is a right linear combination of elements of U_{u_0}, all of which are themselves right linear combinations of elements of U_*. Hence u_0 is a right linear combination of elements of U_*, in contradiction to our hypothesis that $L_r\,U_*$ did not contain u_0.

3. U_* is right linearly independent, for, if not,

$$\exists\, ((u_1 \cdots u_n)^{\text{distinct } U_*}\, a_1 \neq 0 \cdots a_n \neq 0) \,.\ni.\, \sum_i u_i a_i = 0_U,$$

which is in contradiction to the definition of U_*.

THEOREM 3. *If there exists a normally ordered right base U_* for U, then there exists a normally ordered proper right base U_{0*} for every right linear subset $U_0 \neq [0_U]$ of U.*

Proof. By reasoning analogous to that used in the proof of Theorem 2 we see that $[\text{all } u_* \,.\ni.\, u_*$ is right linearly independent as to $U_{*u_*}] = U'$ is a proper right base for U.

From § 2, Theorem 5, it follows that every u is expressible uniquely in the form $u = \sum_i u'_i a_i$, where $a_i \neq 0\,(i)$ and $i < j\,.).\, u'_i$ precedes u'_j.

Consider $U_1 = [\text{all } u_0 \,.\ni.\, a_1 \text{ (for } u_0) = 1]$; obviously $L_r\,U_1 = U_0$.

Consider the class $[U'_p]$ of all sets U'_p consisting of a finite number of elements of U'. We can normally order this class as follows:

U'_1 precedes U'_2 if

(1) U'_1 has fewer members than U'_2;

(2) U'_1 has the same number of elements as U'_2, but the first member of U'_1 not in U'_2 precedes the first member of U'_2 not in U'_1.

Corresponding to every set U'_p there exists a class U_{1p} consisting of all elements of U_1 which are right linear combinations with non-zero coefficients of all elements of U'_p, and no other elements. The sets U_{1p} (i. e., the non-vacuous classes U_{1p}) are in 1-1 correspondence with the sets U'_p from which they arise. Every element u_1 of U_1 falls in one and only one such set. We say that the set U_{11} precedes U_{12} provided U'_1 precedes U'_2.

No element u'_0 is right linearly independent of the elements in the sets preceding the set to which it belongs unless it is the only element in this set. For consider a set U_{11} containing two distinct elements u_{111} and u_{112} where

$$u_{111} = u_{11} + \sum_i^n u_{1i}\, a_{1i}, \tag{1}$$

$$u_{112} = u_{11} + \sum_i^n u_{1i}\, a_{2i}, \tag{2}$$

where n is the number of elements in U' and no multiplier a_{1i} or a_{2i} is zero. Hence $L_r[u_{111}, u_{112}]$ contains $u_{111} - u_{112} = u_{03}$ and $u_{03} \neq 0$, and hence there exists a number a such that $u_{03}\, a$ belongs to U_1 and is in a set which precedes U_{11}. Moreover there exists a number a_1 such that $u_{111} - u_{03}\, a_1 = u_{04}$ is in a set preceding U_{11}. However, it is evident that $L_r[u_{03}\, a, u_{04}]$ contains both u_{111} and u_{112}.

Denote by U_2 the totality of elements of U_1 which are the only members of the sets to which they respectively belong. These elements may be normally ordered as were the sets to which they belonged and constitute a right base for U_0, and hence by reasoning analogous to that used in the proof of Theorem 2 it can be readily shown that [all u_2 :∋: u_2 is right linearly independent as to U_{2u_2}] is a normally ordered proper right base for U_0.

THEOREM 4. *If a subset U_0 of U is right linear and if U_1 and U_2 are normally ordered proper right bases for U_0, then U_1 and U_2 have the same cardinal number.*

Proof. For every u_1 consider

U_{3u_1} = [all u_2 right linearly independent as to $\mathbf{U}\,[U_{2u_2}, U_{1u_1}]$],

U_{4u_1} = [all u_2 right linearly independent as to $\mathbf{U}\,[U_{2u_2}, U_{1u_1}, u_1]$];

obviously, for every u_1, $U_{3u_1} \supset U_{4u_1}$ and $L_r \cup [U_{1u_1} U_{3u_1}] = U_0$, but $L_r U_{1u_1}$ does not contain the element u_1. Hence every element u_1 of U_1 is a right linear combination with non-zero coefficients of at least one element of U_{3u_1} and elements of U_{1u_1}. Therefore U_{3u_1} contains at least one element not in U_{4u_1}. Let u_1 correspond to the first u_2 belonging to U_{3u_1} but not in U_{4u_1}.

Moreover, it is obvious that

$$u_{11} \sim u_{21} \,.\, u_{12} \sim u_{22} \,.\, u_{11} \neq u_{12} :): u_{21} \neq u_{22},$$

and hence there exists a one to one correspondence between U_1 and a part of U_2 and similarly between U_2 and a part of U_1.

THEOREM 5. *If a right linear subset U_0 of U has a normally ordered proper right base U_1, then any other proper right base U_2 for U_0 can be normally ordered.*

Proof. Consider the class $[U_{1p}]$ of all finite sets of elements of U_1. This class may be normally ordered. Corresponding to every such set U_{1p} consider the totality U_{2p} of elements of U_2 which are right linear combinations with non-zero coefficients of all the elements of U_{1p}. Every element of U_2 falls in one and only one such set. Since U_2 is right linearly independent the number of elements in any set U_{2p} can not exceed the number of elements in U_{1p}. Hence U_2 consists of the members of a normally orderable class of finite sets of elements and is therefore normally orderable.

Definitions of *right* (*left*) *rank.* If a subset U_0 of U is right linear and there exists a normally ordered proper right base for U_0 we say that the cardinal number of such a right base is the right rank of U_0: $(rk_r(U_0))$.

THEOREM 6. *If the subsets U_1 and U_2 of U are right linear, have normally ordered right bases, and $U_1 \supset U_2$, then*

$$rk_r(U_1) \geqq rk_r(U_2).$$

COROLLARY. *Relative to a subset U_0 of U of such a nature that there exists a normally ordered right base for LU_0*

$$rk_r(LU_0) \geqq rk_r(L_r U_0) \geqq rk_r(L_0 L_r U_0) \geqq rk_r(L_0 U_0).$$

Note: According to Theorem 3 all of these ranks exist.

THEOREM 7. *If there exists a normally ordered right base U_* for U, then relative to a right linear subset U_0 of U there exists a right linear subset U_1 of U such that U_0 and U_1 are supplementary.*

Proof. Case 1. $U_0 = U$. In this case $[0_U]$ is effective as U_1 of the theorem.

Case 2. $U_0 \neq U$. Consider $U_{1*} = [\text{all } u_* \ .\ni.\ u_*$ is right linearly independent as to $[U_{*_{u_*}} U_0]]$; $L_r U_{1*}$ is effective as U_1 of the theorem.

1. By proof analogous to Theorem 2 and by § 3, Theorem 3, we see that $U = L_r \mathbf{U} [U_{1*} U_0] = L_r U_{1*} + L_r U_0 = L_r U_{1*} + U_0$;

2. $\mathbf{\cap} [U_0 L_r, U_{1*}]$ contains only 0_U, for if it contained an element u_0 of U_0 such that $u_0 \neq 0_U$, then u_0 would be a right linear combination with nonzero coefficients of elements of U_{1*} and hence there would exist an element of U_{1*} not right linearly independent as to U_0 and the preceding elements of U_{1*}.

Definition of a ***difference set.***† Relative to a properly linear subset U_0 of U we define the *difference* of U and U_0 $(U - U_0)$ as follows:

$$U - U_0 = [\text{all } U_1 :\ni: u \text{ belongs to } U_1 .).\ [u] + U_0 = U_1 \equiv \{u\}].$$

It is seen that $U - U_0$ is not itself a set of elements of U but a class of sets of elements such that the members of any one set differ from each other by an element of U_0 and all elements of U which differ by an element of U_0 belong to the same set.

LEMMA 1.

$$U_0^l :): \{u_1\} = \{u_2\} .).\ (1)\ \{u_1 a\} = \{u_2 a\} = \{u_1\} a \quad (a);$$
$$(2)\ \{a u_1\} = \{a u_2\} = a\{u_1\} \quad (a).$$

LEMMA 2.

$$U_0^l .:):. \{u_1\} = \{u_2\} . \{u_3\} = \{u_4\}$$
$$:): \{u_1 + u_3\} = \{u_2 + u_4\} = \{u_1\} + \{u_3\}.$$

From the lemmas and definitions above it follows at once that relative to a properly linear subset U_0 of U where $U_0 \neq U$ the difference, $U - U_0$, together with the number system $\mathfrak{A}$ and with addition defined as in Lemma 2 and multiplication as in Lemma 1, forms a system Σ satisfying the postulates of I, § 1.

THEOREM 8. *If two properly linear subsets U_0 and U_1 of U are supplementary then U_1 is isomorphic with $U - U_0$ under the correspondence $u_1 \sim \{u_1\}$ (u_1).*

Proof. 1. $u_{11} \neq u_{12} .).\ \{u_{11}\} \neq \{u_{12}\}$, for if not $(u_{11} - u_{12})$ would belong both to U_0 and U_1, which is impossible since U_0 and U_1 are supplementary.

2. In every set $\{u\}$ there exists an element of U_1, for $\exists\ u_0, u_1 .\ni.\ u = u_0 + u_1$ and therefore $\{u\} = \{u_1\}$.

† For an abstract definition of a difference algebra see L. E. Dickson, *Algebras and their Arithmetics*, p. 36 ff. Our definition could be made more general by not requiring that U_0 be properly linear, but many of the most useful properties would not be preserved. We therefore limit ourselves to this case.

3. The preservation of the correspondence under addition and multiplication follows from the fact that U_0 and U_1 are properly linear and from Lemmas 1 and 2.

5. **Systems with a commutative base.** In this section we will consider a system Σ of such a nature that there exists a proper right base U_* of U which is commutative with $\mathfrak{A}$; hence U_* is a proper left base for U. We show that in this case U is isomorphic with the set of all finitely non-zero vectors on a certain range P and that any properly linear subset U_0 of U has a commutative right base and conversely.

If Σ is such that U has a commutative proper right base then we say that Σ is of type 1.

Note. That not all systems Σ are of type 1 is seen from the following example.

Consider both $\mathfrak{A}$ and U as the Hilbert example† of a Veronesean number system. Associate with every number $a \equiv (a(i)\ (i = -\infty \cdots +\infty))$ the number a' where $a' \equiv (a'(2i) = a(i) \,.\, a'(2i+1) = 0\ (i = -\infty \cdots +\infty))$. We define the processes $\oplus$, $\odot_r$ and $\odot_l$ as follows:

$$u_1 u_2 .) .\ u_1 \oplus u_2 \equiv u_1 + u_2,$$

$$a\, u\ .) .\ u \odot_r a \equiv u a,$$

$$a\, u\ .) .\ a \odot_l u \equiv u \odot_r a' = u a',$$

where the addition and multiplication in the right hand members of the above definitions are the ordinary addition and multiplication for numbers of such a system. Then $0_U (= 0)$ is the only element of U which is commutative with every number of $\mathfrak{A}$, for consider $a_1 \equiv (a_1(1) = 1, a_1(i) = 0\ (i \neq 1))$; then $a_1' = (a_1'(2) = 1, a_1'(i) = 0\ (i \neq 2))$, and it follows that $u \neq 0_U .) .\ a_1 \odot_l u = u a_1' \neq u a_1 = u \odot_r a_1$. This is also an example of a properly linear set with a right rank different from the left rank, for $u_0 \equiv (u_0(0) = 1 \,.\, u_0(i) = 0\ (i \neq 0))$ is a proper right base for U; but U has left rank 2 for u_0 and $u_1 \equiv (u_1(1) = 1 \,.\, u_1(i) = 0\ (i \neq 1))$ form a proper left base for U, for $a u_0 = u .) .\ u(i) = 0$ for i odd and $a u_1 = u .) .\ u(i) = 0$ for i even.

Theorem 1. *If Σ is of type 1 and if we consider $P \equiv [p] \equiv U_*$ and $U' \equiv$ [all vectors u' on P to $\mathfrak{A}$ finitely non-zero] and $U'_* \equiv [\delta_p\ (p)]$ where $\delta_p(p) = 1$ and $\delta_p(p_1) = 0$ for every $p_1 \neq p$, then U is isomorphic with U' under the correspondence*

$$u = \sum_i^{1,n} u_{*i} a_i \sim u' = \sum_i^{1,n} \delta_{u_{*i}} a.$$

† Ex. 6 of a system $\mathfrak{A}$ of type A.

Proof. The theorem follows at once from the fact that U_* is commutative.

Relative to a general range P and any number system $\mathfrak{A}$ of type A the class of all vectors u on P to $\mathfrak{A}$ finitely non-zero is a set U belonging to a system Σ of type 1. In the remainder of this article we will therefore consider only systems of finitely non-zero vectors on a range P.

Relative to P^1 and the numbers of $\mathfrak{A}$ as vectors on P^1 to $\mathfrak{A}$ obviously $a \neq 0$.). $L(a) = L(1)$.

LEMMA 1. *If P is finite and u is a vector on P to $\mathfrak{A}$ nowhere zero, then either there exist a vector u_1 and a number $a \neq 0$ such that u_1 is commutative and $u_1 a = u$ or there exist in $L(u)$ two vectors u_2 and u_3 and elements p_2 and p_3 of the range P such that $u_2(p_2) = 0$, $u_3(p_3) = 0$ and $L(u) = L[u_2 u_3]$.*

Proof. It is sufficient to prove this for the special case $P = P^2$. Then $u = (a_1, a_2)$ with $a_1 a_2 \neq 0$ and $L(u) = L(u_1)$ where $u_1 = (1, a_2 a_1^{-1})$. If u is not commutative, $\exists\, a \;.\ni.\; a a_2 a_1^{-1} a^{-1} \neq a_2 a_1^{-1}$, and since Lu contains $(1, a a_2 a_1^{-1} a^{-1})$ it also contains $(0, a_2 a_1^{-1} - a a_2 a_1^{-1} a^{-1})$; therefore Lu contains $(0, 1)$ and $(1, 0)$.

LEMMA 2. *Relative to a general range P and the set U of all vectors u on P to $\mathfrak{A}$ finitely non-zero, it is true that u :): $\exists\, U_0^c \;.\ni.\; L U_0 = L u$.*

Proof. This lemma follows by the repeated application of Lemma 1.

Theorem 2 follows at once from Lemma 2.

THEOREM 2. *Relative to a general range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, there exists a commutative right base U_{0*} for U_0 which therefore is also a left base for U_0.*

THEOREM 3. *Relative to a normally ordered range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, there exists a normally ordered commutative proper right base U_{0*} for U_0 which therefore is also a proper left base for U_0.*

Proof. Since by Theorem 2 U and U_0 are the linear extensions respectively of their commutative subsets U' and U_0', it follows from I, § 4, Theorem 1, and I, § 2, Theorem 6, that we need only prove the theorem relative to the system $\Sigma_1 = (\mathfrak{A}' U' \oplus \odot_r \odot_l)$. In this form, however, the theorem is merely a special case of I, § 4, Theorem 3.

Relative to a normally ordered range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, the right rank of U_0 is equal to the left rank of U_0. In this case we will speak of either the right or left rank of U_0 as the *rank* of U_0 ($rk\, U_0$).

II. Sets of vectors on a finite range

Contents

Introduction. In this section we will consider a system composed of a number system $\mathfrak{A}$ of type A, the totality $V = [v]$ of all vectors on a finite range $P^n = [1, 2, 3, \cdots, n]$ to $\mathfrak{A}$ and addition and multiplication defined as in Example 2 of I, § 1, for a system Σ. Thus we are dealing with a special case of a system Σ of type 1.

We introduce notations as follows for matrices, vectors and their composition:

$$W = [\text{all matrices } w \text{ on } PP \text{ to } \mathfrak{A}]$$

$$w = w(i,j) \qquad (i = 1, \cdots, n, j = 1, \cdots, n).$$

We say that a matrix w is commutative, w^c, in case every element of w belongs to $\mathfrak{A}'$.

*Composition of matrices: S notation:**

$$w_3 = Sw_1w_2 :=: w_3(j,k) = \sum_i w_1(j,i)w_2(i,k) \qquad (j,k).$$

Composition of a vector and a matrix.

$$v_1 = Swv :=: v_1(i) = \sum_j^{1,n} w(i,j)\,v(j) \qquad (i).$$

$$v_1 = Svw :=: v_1(i) = \sum_j^{1,n} v(j)\,w(j,i) \qquad (i).$$

Inner product of two vectors:

$$a = Sv_1v_2 :=: a = \sum_i^{1,n} v_1(i)\,v_2(i).$$

We say that a matrix w is non-singular (w^{ns}) in case it has a right and left reciprocal, which is equivalent to the rows of w being left linearly independent and the columns right linearly independent. We will use the notation δ for the identity matrix.

* This notation is that used by E. H. Moore in his course in General Analysis.

1. Normal forms for bases. In this article we define what we mean by the base of a right (left) linear set being in semi-normal or normal form. We show that two right linear sets are equal if and only if the normal forms of their bases are equal. We show also that a right base for a properly linear set which is in semi-normal form is also a left base for the same set and is composed of commutative vectors.

THEOREM 1. *If a right linear subset V_0 of V has right rank r, and σ is a set of distinct elements $p_1, \cdots, p_r$ of the range P such that V as on σ is of right rank r, then their exists one and only one set of vectors $V_{0\sigma} = (v_{01} \cdots v_{0r})$ of such a nature that*

$$(1) \qquad L_r V_{0\sigma} = V_0;$$

$$(2) \qquad v_{0i}(p_i) = 1 \qquad (i = 1, \cdots, r);$$

$$(3) \qquad v_{0i}(p_j) = 0 \qquad (i \neq j,\ i = 1, \cdots, r, j = 1, \cdots, r).$$

Proof. 1. $\exists\, V_1 = (v_1 \cdots v_r) :\ni: V_0 \supset V_1$ and on σ is identical with the set $\delta_{p_1}, \cdots, \delta_{p_r}$. V_1 as on σ has right rank r. Therefore aside from uniqueness V_1 is effective as the $V_{0\sigma}$ of the theorem.

2. $V_{0\sigma}$ is unique, for consider $i \leqq r$ and v'_i in V_0 of such a nature that $v'_i(p_i) = 1$ and $v'_i(p_j) = 0$ for $j \leqq r$ and unequal to i; hence $v'_i = v_{0i}$ since it belongs to V_0 and therefore to $L_r V_{0\sigma}$.

THEOREM 2.

$V_0^{rl} \cdot rk_r V_0 = r ::) :: \exists \mid \sigma_* = (p_1 \cdots p_r) .:\ni:.$

(1) V_0 *as on* σ_* *is of right rank* r;

(2) $p_1 < p_2 \cdots < p_r$;

(3) $\sigma' = (p'_1 < p'_2 \cdots < p'_r) .\ni. V_0$ *as on* σ' *is of right rank* $r :) : \sigma_* \langle\langle \sigma'$.

The proof is obvious.

We say that a right base V_{01} of a right linear subset V_0 of V is in semi-normal form in case there exists a σ satisfying the conditions of Theorem 1 for which V_{01} is effective as the V_0 of the theorem. In such a case we say that V_{01} is in normal form provided σ is effective as σ_* of Theorem 2.

THEOREM 3. *Two right linear sets are equal if and only if the normal forms of their bases are equal.*

THEOREM 4. *Relative to a properly linear subset V_0 of V a right base V_{01} for V_0 in semi-normal form is also a left base for V_0 in semi-normal form and is commutative.*

Proof. Since $rk_r V_0 = rk_l V_0$, V_{01} is a left base for V_0 in semi-normal form. Hence the right hand multiples of the vectors of V_{01} must be equal to the left hand multiples with the same coefficients and therefore V_{01} is commutative.

COROLLARY. *The normal form of the right base of a properly linear subset V_0 of V is equal to the normal form of its left base and is composed of commutative vectors.*

2. **Orthogonal sets.** In this article we give a definition of the right (left) orthogonality of one vector to another in respect to a commutative non-singular matrix w, and in terms of these relations we define the right (left) orthogonal complements $0_{rw} V_0$ $(0_{lw} V_0)$ of a subset V_0 of V in respect to w. We then study the iteration of the processes 0_{rw}, 0_{lw} together with L_r, L_l, etc. and give the iteration table of the resulting twelve distinct processes, one set of whose generators are 0_{rw}, 0_{lw} and L_0; and show that these are closed under further iteration. We then make a generalization of these processes such that the resulting iteration table is abstractly equivalent to that obtained from 0_{rw}, 0_{lw}, and L_0 and applies to the more general situation of Section I.

The statements we make in the remainder of this article will be relative to a commutative symmetric non-singular matrix w.

We say that v_1 is left orthogonal to v_2 and v_2 is right orthogonal to v_1 in respect to w in case $S^2 v_1 w v_2 = 0$. We define the *right (left) orthogonal complement* of a subset V_0 of V in respect to w, $0_{rw} V_0$ $(0_{lw} V_0)$, and the sets $0_w V_0$ and $0_{0w} V_0$ as follows:

$$0_{rw} V_0 = [\text{all } v :\ni: v_0 .). S^2 v_0 w v = 0].$$

$$0_{lw} V_0 = [\text{all } v :\ni: v_0 .). S^2 v w v_0 = 0].$$

$$0_w V_0 = 0_{rw} L V_0 = 0_{lw} L V_0 \qquad \text{(see Lemma 3).}$$

$$0_{0w} V_0 = 0_{rw} L_0 V_0 = 0_{lw} L_0 V_0 \qquad \text{(see Lemma 3).}$$

In case $w = \delta$ the orthogonality condition reduces to the vanishing of the inner product $S v_1 v_2$, and we say v_1 is left orthogonal to v_2 etc., and we use the notations $0_r V_0$ for $0_{r\delta} V_0$ etc.

LEMMA 1. V_0 .). (1) $0_{lw} V_0 = 0_{lw} L_r V_0$.
(2) $0_{rw} V_0 = 0_{rw} L_l V_0$.

LEMMA 2. $V_1 \supset V_2$.). (1) $0_{rw} V_2 \supset 0_{rw} V_1$,
(2) $0_{lw} V_2 \supset 0_{lw} V_1$.

LEMMA 3. V_0^l .). $0_{lw} V_0 = 0_{rw} V_0$.

Proof. There exists a commutative base V_{01} for V_0 and hence $0_{lw} V_0 = 0_{lw} V_{01} = 0_{rw} V_{01} = 0_{rw} V_0$.

LEMMA 4. V_0 :): $(0_{rw} V_0)^{rl} . (0_{lw} V_0)^{ll}$.

LEMMA 5.† V_0 .). (1) $0_{rw} 0_{lw} V_0 = L_r V_0$.
(2) $0_{lw} 0_{rw} V_0 = L_l V_0$.

Proof: Obviously $0_{rw} 0_{lw} V_0 \supset L_r V_0$, and $0_{rw} 0_{lw} V_0$ is right linear, hence the lemma is true provided $rk_r 0_{rw} 0_{lw} V_0 = rk_r L_r V_0$. This follows if we show that

(I) V_0 .). (1) $rk_l 0_{lw} V_0 + rk_r L_r V_0 = n$.
(2) $rk_r 0_{rw} V_0 + rk_l L_l V_0 = n$.

However, since w is non-singular we need only prove (I) for the special case in which $w = \delta$. Let $r = rk_r L_r V_0$. Consider $(p_1 \cdots p_r) \equiv \sigma$ as the effective σ_* of § 1, Theorem 2, for $L_r V_0$, and $(v_1 \cdots v_r) \equiv$ the normal form of the right base for $L_r V_0$. Let $\sigma' \equiv (p'_1 \cdots p'_{n-r})$ be the set of elements of P not in σ. Then consider $V'_* \equiv (v'_1 \cdots v'_{n-r})$, where, for every i, $v'_i(p'_i) = 1$, $v'_i(p'_j) = 0$ $(i \neq j)$, $v'_i(p_k) = -v_k(p'_i)$ $(k = 1 \cdots r)$, V'_* is a left base in semi-normal form for the left linear set $V' = L_l V'_*$. which is of left rank $n-r$. Moreover, $0_l V_0 \supset V'$. Hence the $rk_l 0_l V_0 \geq n-r$. However, if $rk_l 0_l V_0 > n-r$ there would exist a vector v in $0_l V_0$ and an element p_i $(i \leq r)$ of the range P such that $v(p_i) = 1$ and $v(p'_j) = 0$ for every $j \leq n-r$. Since this is impossible, $rk_l 0_l V_0 = n-r$ and our lemma is proved.

LEMMA 6. V_0 :): $(0_w V_0)^l . (0_{0w} V_0)^l$.

This follows directly from Lemmas 3 and 4.

LEMMA 7. V_0^l .). $L_r V_0 = L_l V_0 = L V_0 = L_0 V_0 = L_0 L_r V_0 = L_0 L_l V_0$.

LEMMA 8. V_0 .). $0_{rw} L_r V_0 = 0_{rw} L V_0 = 0_{lw} L V_0 = 0_{lw} L_l V_0 = 0_w V_0$.

LEMMA 9. V_0^{rl} .). $(0_{rw} V_0)^l$ and V_0^{ll} .). $(0_{lw} V_0)^l$.

LEMMA 10. V_0 :): $(0^2_{rw} V_0)^l . (0^2_{lw} V_0)^l$.

LEMMA 11. V_0 .). $L_0 0_{rw} V_0 = 0_w V_0 = L_0 0_{lw} V_0$.

Proof. $L V_0 \supset L_r V_0$ and hence $0_w V_0 = 0_{lw} L V_0 \subset 0_{lw} L_r V_0 = 0_{lw} V_0$, and since $0_w V_0$ is properly linear it follows that

(1) $$0_w V_0 \subset L_0 0_{lw} V_0.$$

† This lemma in its equivalent matricial form is due to E. H. Moore and is given in his course in General Analysis. The proof is the writer's. It may be stated in the following form:

$$P' P'' w''' v'_1 .:):. \exists\, v'' .\ni. S'' w''' v'' = v'_1 :\sim: S' v' w''' = 0_{V'} .). S' v' v'_1 = 0.$$

$L_0\, L_l\, V_0 \subset L_l\, V_0$ and hence $0_w\, L_0\, L_l\, V_0 = 0_{rw}\, L_0\, L_l\, V_0 \supset 0_{rw}\, L_l\, V_0 = 0_{rw} V_0$, and since $0_w\, L_0\, L_l V_0$ is properly linear it follows that $0_w\, L_0\, L_l\, V_0 \supset L\, 0_{rw} V_0$, and hence

$$(2) \qquad 0_w^2\, L_0\, L_l\, V_0 = L_0\, L_l\, V_0 \subset 0_w\, L\, 0_{rw}\, V_0;$$

using Lemma 5 and $0_{lw}\, V_0$ as V_0 in (2) we obtain

$$(3) \qquad L_0\, 0_{lw}\, V_0 \subset 0_w\, L 0_{rw}\, 0_{lw}\, V_0 = 0_w\, L\, L_r\, V_0 = 0_w\, L\, V_0 = 0_w\, V_0,$$

hence by (1) and (3) we have $0_w\, V_0 = L_0\, 0_{lw}\, V_0$.

LEMMA 12. $V_0\; .\!).\; L\, 0_{rw}\, V_0 = 0_{0w}\, L_l\, V_0$.

Proof. Applying Lemmas 11 and 1 to $0_{rw}\, V_0$ for V_0 we obtain $L_0\, L_l\, V_0 = L_0\, 0_{lw}\, 0_{rw}\, V_0 = 0_w\, 0_{rw}\, V_0 = 0_w\, L\, 0_{rw}\, V_0$ and the lemma follows at once. This may also be stated in the form

$$(12_1) \qquad V_0\; .\!).\; 0_w\, 0_{rw}\, V_0 = L_0\, L_l\, V_0.$$

By use of the above lemmas we derive Table I, which gives the results of iteration of the processes $L_r, L_l, \cdots, 0_{0w}$. Thus we find from row 4, column 7, that $V_0\; .\!).\; L_0\, 0_{rw}\, V_0 = 0_w\, V_0$. It should be especially noted that the three processes $0_{rw}, 0_{lw}$, and L_0 are generators of the whole table.

TABLE I

	L_r	L_l	L	L_0	$L_0\, L_r$	$L_0\, L_l$	0_{rw}	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_r	L_r	L	L	L_0	$L_0\, L_r$	$L_0\, L_l$	0_{rw}	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_l	L	L_l	L	L_0	$L_0\, L_r$	$L_0\, L_l$	$L0_{rw}$	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L	L	L	L	L_0	$L_0\, L_r$	$L_0\, L_l$	$L0_{rw}$	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_0	$L_0\, L_r$	$L_0\, L_l$	L	L_0	$L_0\, L_r$	$L_0\, L_l$	0_w	0_w	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
$L_0\, L_r$	$L_0\, L_r$	L	L	L_0	$L_0\, L_r$	$L_0\, L_l$	0_w	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
$L_0\, L_l$	L	$L_0\, L$	L	L_0	$L_0\, L_r$	$L_0\, L_l$	$L0_{rw}$	0_w	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
0_{rw}	0_w	0_{rw}	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0\, L_l$	L_r	$L_0\, L_l$	$L_0\, L_r$	L	L_0
0_{lw}	0_{lw}	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L_l	$L_0\, L_r$	$L_0\, L_l$	$L_0\, L_r$	L	L_0
$L0_{rw}$	0_w	$L0_{rw}$	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0\, L_l$	L	$L_0\, L_l$	$L_0\, L_r$	L	L_0
$L0_{lw}$	$L0_{lw}$	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L	$L_0\, L_r$	$L_0\, L_l$	$L_0\, L_r$	L	L_0
0_w	0_w	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0\, L_l$	$L_0\, L_r$	$L_0\, L_l$	$L_0\, L_r$	L	L_0
0_{0w}	$L0_{lw}$	$L0_{rw}$	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L	L	$L_0\, L_l$	$L_0\, L_r$	L	L_0

The proof of Table I may be readily effected by using the lemmas as listed in the following table. Numbers refer to lemmas of this article.

	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	0_{rw}	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_r	I § 3 Table I						4	4	6, 7, 8, 12			
L_l							4	4				
L							Def.	Def.				
L_0							11	11				
$L_0 L_r$							11	4				
$L_0 L_l$							4	11				
0_{rw}	8	1	7, 8, 12_1				4, 12_1	5				
0_{lw}	1	8					5	4, 12_1				
$L0_{rw}$	6	1					4, 12_1	5				
$L0_{lw}$	1	6					5	4, 12_1				
0_w	Def.	Def.					12_1	12_1				
0_{0w}	12	12					11	11				

In order to see that there exists a number system $\mathfrak{A}$, a finite range P, a commutative symmetric non-singular matrix w on PP to $\mathfrak{A}$ and a set V_0 of vectors on P to $\mathfrak{A}$ such that the twelve sets $L_r V_0, \cdots, 0_{0w} V_0$ are distinct, consider $\mathfrak{A} = Q$ (real quaternions), the range P^5 and $w \equiv \delta$ and

$$V_0 = \begin{matrix} (1 & 0 & 0 & 0 & 0\ ) \\ (0 & 1 & i & 0 & 0\ ) \\ (0 & j & k & 0 & 0\ ) \\ (0 & 0 & 0 & 1 & i\ ) \\ (0 & 0 & 0 & j & -k). \end{matrix}$$

The twelve sets $L_r V_0, \cdots, 0_{0w} V_0$ are given below with their bases in the normal form:

$$L_r V_0 = L_r \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & i & 0 & 0) \\ (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L_l V_0 = L_l \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \\ (0 & 0 & 0 & 1 & i) \end{matrix}$$

$$L V = V \qquad L_0 V_0 = L\,(1\ \ 0\ \ 0\ \ 0\ \ 0)$$

$$L_0 L_r V_0 = L \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L_0 L_l V_0 = L \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \end{matrix}$$

$$0_r V_0 = L_r (0\ \ 0\ \ 0\ \ 1\ \ i) \qquad 0_l V_0 = L_l (0\ \ 1\ \ i\ \ 0\ \ 0)$$

$$L0_r V_0 = L \begin{matrix} (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L0_l V_0 = L \begin{matrix} (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \end{matrix}$$

$$0\,V_0 = 0_V = (0\ 0\ 0\ 0\ 0) \qquad 0_0\,V_0 = L\begin{matrix}(0 & 1 & 0 & 0 & 0)\\(0 & 0 & 1 & 0 & 0)\\(0 & 0 & 0 & 1 & 0)\\(0 & 0 & 0 & 0 & 1)\end{matrix}$$

The iteration of the processes represented in Table I is associative. This fact can be readily checked by consideration of the generators 0_{rw}, 0_{lw}, L_0.

There are 107 distinct closed sub-tables in Table I. They are listed below by giving the generators, in each case choosing the minimum number necessary. The order chosen is not dependent on the number of generators but on the number of processes generated. This number is shown in Roman numerals. If the two tables include the same number of processes we list first that one whose first process (relative to the order of Table I) not in the second table precedes the first process in the second table but not in the first. We denote by setting two numbers before a list of generators that one may secure by parity a distinct table which therefore is not listed.

I. 1, 2. L_r. 3. L. 4. L_0. 5, 6. $L_0\,L_r$.

II. 7, 8. L_r, L. 9, 10. L_r, $L_0\,L_r$. 11. L, L_0. 12, 13. L, $L_0\,L_r$. 14. 0_w. 15, 16. L_0, $L_0\,L_r$. 17. 0_{0w}. 18. $L_0\,L_r$, $L_0\,L_l$. 19, 20. $L0_{lw}$.

III. 21. L_r, L_l. 22, 23. L_r, L, $L_0\,L_r$. 24, 25. L_r, $L_0\,L_l$. 26, 27. L_r, 0_w. 28, 29. L_r, L_0. 30, 31. L_r, $L0_{lw}$. 32, 33. L, L_0, $L_0\,L_r$. 34. L, $L_0\,L_r$, $L_0\,L_l$. 35. L_0, $L_0\,L_r$, $L_0\,L_l$. 36, 37. 0_{lw}.

IV. 38, 39. L_r, L_l, $L_0\,L_r$. 40. L_r, L_l, 0_w. 41, 42. L_r, L, L_0. 43, 44. L_r, $L_0\,L_r$, $L_0\,L_l$. 45, 46. L_r, 0_{lw}. 47. L, L_0, $L_0\,L_r$, $L_0\,L_l$. 48. L, 0_{0w}. 49, 50. L, $L0_{lw}$. 51, 52. L_0, $L0_{lw}$. 53. $L_0\,L_r$, $L0_{rw}$.

V. 54. L_r, L_l, $L_0\,L_r$, $L_0\,L_l$. 55, 56. L_r, L_0, $L_0\,L_l$. 57, 58. L_r, L, $L0_{lw}$. 59, 60. L_r, $L0_{rw}$. 61, 62. L_r, 0_{0w}. 63, 64. L, 0_{lw}.

VI. 65. L_r, L_l, L_0. 66, 67. L_r, L_l, $L0_{lw}$. 68, 69. L_r, L, 0_{lw}. 70, 71. L_r, 0_{rw}. 72, 73. L, L_0, $L0_{lw}$. 74. L, $L_0\,L_r$, $L0_{rw}$.

VII. 75, 76. L_r, L_l, 0_{lw}. 77, 78. L_r, L, L_0, $L0_{lw}$. 79, 80. L_r, $L_0\,L_r$, $L0_{rw}$. 81, 82. L_0, 0_{lw}. 83, 84. $L_0\,L_r$, 0_{rw}.

VIII. 85. L_r, L_l, $L_0\,L_r$, $L0_{rw}$. 86, 87. L_r, L_0, 0_{lw}. 88, 89. L_r, $L_0\,L_r$, 0_{rw}. 90, 91. L_r, $L_0\,L_l$, 0_{lw}. 92. L, L_0, $L_0\,L_r$, $L0_{rw}$.

IX. 93, 94. L_r, L_l, $L_0 L_r$, 0_{rw}. 95, 96. L_r, L_0, $L0_{rw}$. 97, 98. L_0, $L_0\,L_r$, 0_{rw}.

X. 99. L_r, L_l, L_0, $L0_{rw}$. 100. 0_{rw}, 0_{lw}. 101, 102. L_r, L_0, 0_{rw}. 103, 104. L_r, L_0, $L_0\,L_l$, 0_{lw}.

XI. 105, 106. L_r, L_l, L_0, 0_{rw}.

XII. 107. 0_{rw}, 0_{lw}, L_0.

Since relative to a subset V_0 of V we may determine the sets $0_{rw}\,V_0$, $0_{lw}\,V_0$, $L0_{rw}\,V_0$, $L0_{lw}\,V_0$, $0_w\,V_0$ and $0_{0w}\,V_0$ from the sets $L_l\,V_0$, $L_r\,V_0$, $L_0\,L_l\,V_0$, $L_0\,L_r\,V_0$, $L\,V_0$, $L_0\,V_0$ respectively, and conversely, we see that

the Table II of I, § 3, shows us which of the twelve sets are determined, in general, when any combination of a number of the sets is given.

Although we have defined the orthogonal complement of a set V_0 explicitly our table could be arrived at from a postulational point of view.

Consider a system Σ satisfying the postulates of I, § 1, and further two processes T_r and T_l such that corresponding to every subset U_0 of U there exist two subsets $T_r U_0$ and $T_l U_0$ of U and the four following conditions are satisfied:

(1) $$U_0 .).\ (\mathrm{a})\quad T_r U_0 = T_r L_l U_0,$$
$$(\mathrm{b})\quad T_l U_0 = T_l L_r U_0;$$

(2) $$U_0 .).\ (\mathrm{a})\quad T_r T_l U_0 = L_r U_0,$$
$$(\mathrm{b})\quad T_l T_r U_0 = L_l U_0;$$

(3) $$U_1 \supset U_2 .).\ (\mathrm{a})\quad T_r U_1 \subset T_r U_2,$$
$$(\mathrm{b})\quad T_l U_1 \subset T_l U_2;$$

(4) $$U_0^l .).\ T_r U_0 = T_l U_0.$$

From the above conditions we see that

$$U_0 :):\ (T_r U_0)^{rl} . (T_l U_0)^{ll},$$

for $L_r T_r U_0 = T_r T_l L_r T_r U_0 = T_r T_l T_r U_0 = T_r L_l U_0 = T_r U_0$. We define TU_0 and $T_0 U_0$ as $T_r LU_0$ and $T_r L_0 U_0$ respectively. The necessary lemmas for the construction of an iteration table of the T-processes may be readily derived and a table arrived at in terms of the T's of which Table I is a special case.

3. **Applications to the case where $\mathfrak{A}$ is real, complex or quaternionic.** In case $\mathfrak{A}$ is the real, complex or quaternion number system there exists for every number a its conjugate $\bar{a}$. We define the conjugate of a vector $v = (v(i) \mid i)$ as $\bar{v} = (\overline{v(i)} \mid i)$ and the conjugate of a subset V_0 of V as the totality of the conjugates of the vectors of V_0, in notation $\overline{V_0}$. We readily verify the following statements:

(1) $$V_0 .).\ (\mathrm{a})\quad L_r \overline{V_0} = \overline{L_l V_0},$$
$$(\mathrm{b})\quad L\overline{V_0} = \overline{LV_0},$$
$$(\mathrm{c})\quad L_0 \overline{V_0} = \overline{L_0 V_0}.$$

Since in the case of quaternions every properly linear subset of V has a commutative base it follows that

(2) $$\mathfrak{A} = Q .:):.\ V_0^l :\sim: V_0 \supset v .).\ \overline{V_0} \supset \bar{v}.$$

In such a case the notion of what we shall call conjugate orthogonality proves useful. We define the sets $O'_r V_0$ etc. as follows:

$$
\begin{aligned}
O'_r V_0 &= \overline{O_l V_0} &&= [\text{all } v \;.\mathfrak{s}.\; S v v_0 = 0 \qquad (v_0)];\\
O'_l V_0 &= \overline{O_r V_0} &&= [\text{all } v \;.\mathfrak{s}.\; S v_0 \bar{v} = 0 \qquad (v_0)];\\
O' V_0 &= O'_r L V_0 &&= \overline{O V_0};\\
O'_0 V_0 &= O'_r L_0 V_0 &&= \overline{O_0 V_0}.
\end{aligned}
$$

Since every properly linear subset V_0 of V has a commutative base it follows that

(3) $V_0{}^l .).\quad O'_r V_0 = O'_l V_0 = O' V_0;$

(4) $V_0 \;.).\;$ (a) $O'^2_r V_0 = \overline{O_l \overline{O_l V_0}} = \overline{O_r \overline{O_l V_0}} = L_r V_0,$

(b) $O'_r O'_l V_0 = \overline{O_l \overline{O_r V_0}} = \overline{O_r \overline{O_r V_0}} = L_0 L_l V_0.$

By use of the above definitions and lemmas in connection with Table I of § 2, we arrive at Table I which gives the iteration of the processes O'_r, O'_l and L_0 and the processes which they generate.

TABLE I

	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	O'_r	O'_l	LO'_r	LO'_l	O'	O'_0
L_r	L_r	L	L	L_0	$L_0 L_r$	$L_0 L_l$	O'_r	LO'_l	LO'_r	LO'_l	O'	O'_0
L_l	L	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	LO'_r	O'_l	LO'_r	LO'_l	O'	O'_0
L	L	L	L	L_0	$L_0 L_r$	$L_0 L_l$	LO'_r	LO'_l	LO'_r	LO'_l	O'	O'_0
L_0	$L_0 L_r$	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$	O'	O'	LO'_r	LO'_l	O'	O'_0
$L_0 L_r$	$L_0 L_r$	L	L	L_0	$L_0 L_r$	$L_0 L_l$	O'	LO'_l	LO'_r	LO'_l	O'	O'_0
$L_0 L_l$	L	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$	LO'_r	O'	LO'_r	LO'_l	O'	O'_0
O'_r	O'_r	O'	O'	O'_0	LO'_r	LO'_l	L_r	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
O'_l	O'	O'_l	O'	O'_0	LO'_r	LO'_l	$L_0 L_r$	L_l	$L_0 L_r$	$L_0 L_l$	L	L_0
LO'_r	LO'_r	O'	O'	O'_0	LO'_r	LO'_l	L	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
LO'_l	O'	LO'_l	O'	O'_0	LO'_r	LO'_l	$L_0 L_r$	L	$L_0 L_r$	$L_0 L_l$	L	L_0
O'	O'	O'	O'	O'_0	LO'_r	LO'_l	$L_0 L_r$	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
O'_0	LO'_r	LO'_l	O'	O'_0	LO'_r	LO'_l	L	L	$L_0 L_r$	$L_0 L_l$	L	L_0

The example illustrating the distinctness of the twelve processes of Table I, § 2, may be used to show the fact that the twelve processes of the above table are distinct. In this case,

$$
\begin{aligned}
O'_r V_0 &= L_r (0\ 1\ -i\ 0\ 0), & O'_l V_0 &= L_l (0\ 0\ 0\ 1\ -i),\\
LO'_r V_0 &= L O_l V_0, & LO'_l V_0 &= L O_r V_0,\\
O' V_0 &= O V_0, & O'_0 V_0 &= O_0 V_0.
\end{aligned}
$$

It should be noted that relative to subset V_0 of V we can determine the sets $0'_r V_0$, $0'_l V_0$, $L0'_r V_0$, $L0'_l V_0$, $0' V_0$ and $0'_0 V_0$ from the sets $L_r V_0$, $L_l V_0$, $L_0 L_r V_0$, $L_0 L_l V_0$, $L V_0$ and $L_0 V_0$ respectively, and the converse is true. Thus Table II of I, § 3, shows which of the twelve sets $L_r V_0, \cdots, 0'_0 V_0$ are determined, in general, when any combination of these sets is known.

Except for the $0'_r$ and $0'_l$ rows and columns, Table I can be obtained from Table I of § 2 by the substitution of $L0'_r$, $L0'_l$, $0'$ and $0'_0$ for $L0_{lw}$, $L0_{rw}$, 0_w and 0_{0w} respectively. Hence a list of the closed subtables of Table I not involving either $0'_r$ or $0'_l$ may be obtained by the same substitution in the list of the closed subtables of Table I of § 2 which do not involve 0_{rw} and 0_{lw}. Besides these 73 closed subtables we have the following 18 listed by their generators:

II. 1, 2. $0'_r$.
VI. 3, 4. L, $0'_r$.
VII. 5, 6. L_r, $0'_l$.
VIII. 7, 8. L_0, $0'_r$. 9, 10. $L_0 L_r$, $L_0 L_l$, $0'_r$.
IX. 11, 12. L_r, $L_0 L_r$, $0'_l$.
X. 13. $0'_r$, $0'_l$. 14, 15. L_0, $L_0 L_r$, $0'_l$.
XI. 16, 17. L_r, L_0, $0'_l$.
XII. 18. L_0, $0'_r$, $0'_l$.

We can arrive at a generalization of Table I from a postulational point of view. Consider a system Σ satisfying the conditions of I, § 1, and two processes T'_r and T'_l of such a nature that for every subset U_0 of U there exist two subsets $T'_r U_0$, and $T'_l U_0$ and the following conditions are satisfied:

$$(1) \qquad U_0 .). \text{ (a) } \quad T'_r U_0 = T'_r L_r U_0,$$
$$\text{(b) } \quad T'_l U_0 = T'_l L_l U_0;$$

$$(2) \qquad U_0 .). \text{ (a) } \quad T'^2_r U_0 = L_r U_0,$$
$$\text{(b) } \quad T'^2_l U_0 = L_l L_0;$$

$$(3) \qquad U_1 \supset U_2 .). \text{ (a) } \quad T'_r U_2 \supset T'_r U_1,$$
$$\text{(b) } \quad T'_l U_2 \supset T'_l U_2;$$

$$(4) \qquad U_0^l .). \ T'_r U_0 = T'_l U_0;$$

and we make the following definitions:

$$T' U_0 = T'_r L U_0, \qquad T'_0 U_0 = T'_r L_0 U_0.$$

The necessary lemmas for the proof of Table I in terms of the T's instead of the 0's may be readily derived.

4. Identity matrices for properly linear sets. In this article we arrive at a generalization, relative to properly linear subsets V_0 of V and certain commutative non-singular matrices, of the notion of an identity matrix. Moreover we show that for every properly linear subset V_0 of V there exists a commutative symmetric non-singular matrix w which transforms V_0 into a properly linear subset V_1 of V which is supplementary to its orthogonal complement.

Throughout we prove theorems by proving them for the case where $\mathfrak{A}$ is a field, and noting that due to the existence of a commutative base for every properly linear set the theorem follows from the theorem for the special case.

In the case where $\mathfrak{A}$ is a field the six linear processes L_r etc. and the six orthogonal processes 0_{rw} etc. coincide.

LEMMA 1. *If $\mathfrak{A}$ is a field not modulo 2, V_0 is a linear subset of V with rank r greater than zero, and w is an n by n commutative symmetric non-singular matrix such that $\cap [0_w V_0, V_0]$ is the set consisting of the single vector 0_V, then there exists a vector v_0 of V_0 such that $S^2 v_0 w v_0 \neq 0$.*

Proof. Let $v_{01} \cdots v_{0r}$ be a base for V_0. If for every $i \leq r$, $S^2 v_{0i} w v_{0i} = 0$, there exists an i and a j such that $i \neq j$ and $S^2 v_{0i} w v_{0j} \neq 0$. Since $2 = 1 + 1 \neq 0$ it follows that $S^2(v_{0i} + v_{0j}) w (v_{0i} + v_{0j}) = 2 S^2 v_{0i} w v_{0j} \neq 0$, and hence $v_{0i} + v_{0j}$ is effective as the v_0 of the lemma.

That the lemma need not hold for the case of a field with a modulus 2 is shown by the following example. Consider P^3, $\mathfrak{A}$ = integers modulo 2, $w = \delta$, and

$$V_0 = L \begin{matrix} (1\ 1\ 0) \\ (1\ 0\ 1); \end{matrix}$$

the $0 V_0 = L(1\ 1\ 1)$ and V_0 consists of the four vectors $(0\ 0\ 0)$, $(1\ 1\ 0)$, $(1\ 0\ 1)$, and $(0\ 1\ 1)$, but all the elements of V_0 are self orthogonal.

THEOREM 1. *If $\mathfrak{A}'$ is a field not modulo 2, V_0 is a properly linear subset of V and w is an n by n commutative symmetric non-singular matrix such that $\cap [0_w V_0, V_0]$ is the set consisting of the single vector 0_V, then there exists one and only one n by n matrix ε of such a nature that*

$$(1) \qquad S \varepsilon v_0 = v_0 = S v_0 \breve{\varepsilon} \qquad (v_0),$$

$$(2) \qquad V_0 \supset S \varepsilon v \qquad (v),$$

$$(3) \qquad 0_w V_0 \supset (v - S \varepsilon v) \qquad (v).$$

Moreover ε is commutative and therefore $S \varepsilon v = S v \breve{\varepsilon}$ for every v.

Proof. Since V_0 has a commutative base it is sufficient to prove the theorem for the case in which $\mathfrak{A}$ is a field. Relative to two vectors v_1 and v_2 we define the dyad $(v_1 v_2)$ as the matrix Φ where $\Phi(i,j) = v_1(i) v_2(j)\ (i,j)$.

The proof will be divided into two parts, Part 1 the existence of ε and Part 2 the uniqueness of ε.

Part 1. Existence. *Case* 1. $rk\, V_0 = 0$.

In this case the zero matrix is effective as ε.

Case 2. $rk\, V_0 = r > 0$.

According to Lemma 1 there exists v_{01} such that $S^2 v_{01}\, w\, v_{01} \neq 0$. Let

$$\varepsilon_1 = \frac{S(v_{01}\, v_{01})\, w}{S^2 v_{01}\, w\, v_{01}}.$$

and $V_{01} = L(v_{01})\,.\,\varepsilon_1$ is effective as ε for V_{01}. Consider $V_{02} = [v_0 - S\varepsilon_1 v_0\ (v_0)]$. It follows at once that V_{02} is linear and $V_{01} + V_{02} = V_0\,.\,\cap [V_{01}, V_{02}]$ is the set consisting of the single vector 0_V and $0_w V_{02} \supset V_{01}$. Hence if there exists a matrix ε_2 effective as ε for V_{02}, $\varepsilon_1 + \varepsilon_2$ is effective as ε. Thus the existential part of our theorem is true for the case when the rank of V_0 is r in case it is true when the rank of V_0 is $r-1$. Hence, since we have found an effective ε in case $rk\, V_0 = 0$, there exists an effective ε for the case in which $rk\, V_0 = r$.

Part 2. Uniqueness.

Consider ε' and ε'' effective as ε of the theorem:

$$(1)\qquad \begin{aligned} v \cdot v_0 :)\!: S^2(v - S\varepsilon' v)\, w\, v_0 &= 0\,.\, S^2(v - S\varepsilon'' v)\, w\, v_0 = 0 \\ :)\!: S^3(\varepsilon' - \varepsilon'')\, v\, w\, v_0 &= 0; \end{aligned}$$

$$(2)\qquad v\ .).\ S(\varepsilon' - \varepsilon'')\, v \subset V_0.$$

Hence from (1) and (2) and the hypothesis of the theorem it follows that

$$v\ .).\ S(\varepsilon' - \varepsilon'')\, v = 0_V,$$

and hence $\varepsilon' - \varepsilon''$ is the zero matrix and $\varepsilon' = \varepsilon''$.

THEOREM 2. *Relative to a properly linear subset V_0 of V there exists a commutative non-singular n by n matrix Φ such that*

$$w = S\breve{\Phi}\Phi\ .).\ \cap\, [0_w V_0, V_0] = [0_V].$$

Proof. Since V_0 has a commutative base it is sufficient to prove the theorem for the case in which $\mathfrak{A}$ is a field.

Case 1. $\mathfrak{A}$ is a field not modulo 2 or 3.

Consider $V_{0*} = [v_1 \cdots v_r]$ the base for V_0 of normal form. It is readily seen since order is not involved that we may assume $\sigma_* = (1 \cdots r)$.

Let $\Phi_1 \equiv \delta$ if $Sv_1 v_1 \neq 0$ and $\Phi_1 \equiv \delta + \delta_{11}$ if $Sv_1 v_1 = 0$ and $w_1 = S\breve{\Phi}_1 \Phi_1$. Then

$$Sv_1 v_1 \neq 0 \,.).\, Sv_1 w_1 v_1 = Sv_1 v_1,$$
$$Sv_1 v_1 = 0 \,.).\, Sv_1 w_1 v_1 = 3 \neq 0.$$

Let $V_1 = L(v_1)$ and ϵ_1 be the identity matrix of Theorem 2 for V_1 in respect to w_1. Then the set $[v_i - S\epsilon_1 v_i \equiv v_{i1}\ (i = 2, 3, \cdots, r)]$ is a base in semi-normal form for the w_1-orthogonal complement of V_1 in V_0.

This process may be repeated by the general recursion formulas for $j = 1, \cdots, r-1$. (Let $v_{10} = v_1$.)

$V_j = L[v_1, v_{21}, \cdots, v_{j\,j-1}]$;

$\epsilon_j =$ the identity matrix of Theorem 2 for V_j in respect to w_j.

$[v_i - S\epsilon_j v_i = v_{ij}\ (i = j+1, \cdots, r)]$ is a base in semi-normal form for the w_j-orthogonal complement of V_j in V_0, and

$$\Phi_{j+1} = \Phi_j \quad \text{if} \quad Sv_{j+1\,j}\, w_j v_{j+1\,j} \neq 0,$$

but

$$\Phi_{j+1} = \Phi_j + \delta_{j+1\,j+1} \quad \text{if} \quad Sv_{j+1\,j}\, w_j v_{j+1\,j} = 0$$

and

$$w_{j+1} = S\breve{\Phi}_{j+1}\Phi_{j+1}.$$

Hence

$$Sv_{i\,i-1}\, w_{j+1}\, v_{i\,i-1} = Sv_{i\,i-1}\, w_i\, v_{i\,i-1} \neq 0 \qquad (i = 1, \cdots, j),$$
$$Sv_{i\,i-1}\, w_{j+1}\, v_{k\,k-1} = 0 \quad (i \neq k,\ i = 1, \cdots, j,\ k = 1, \cdots, j+1),$$
$$Sv_{j+1\,j}\, w_{j+1}\, v_{j+1\,j} \neq 0.$$

Hence Φ_r is effective as the Φ of the theorem.

Case 2. $\mathfrak{A}$ is a field modulo 2 or 3.

In this case we make $\Phi_{j+1} = \Phi_j + \delta_{j+2\,j+1}$ if $Sv_{j+1\,j}\, w_j\, v_{j+1\,j} = 0$. Otherwise the proof is analogous to that for Case 1.

We may state Theorem 3 as follows: *Relative to a properly linear subset V_0 of V there exists an n by n commutative non-singular matrix Φ which transforms V_0 into a properly linear subset V_0' of such a nature that $0V_0'$ is supplementary to V_0'.*

University of Chicago,
Chicago, Ill.

ON THE REPRESENTATION OF A CERTAIN FUNDAMENTAL LAW OF PROBABILITY*

BY

H. L. RIETZ

1. **Introduction.** In his *Théorie Analytique des Probabilités*,† Laplace makes use of a function which we may write in the form

$$(1)\quad f(x) = \frac{1}{a^n(n-1)!}\left[x^{n-1} - \binom{n}{1}(x-a)^{n-1} + \cdots + (-1)^n(x-na)^{n-1}\right]$$

in which each expression $(x-ma)^{n-1}$, $m = 0, 1, 2, \cdots, n$, is assigned the value zero if the number $x-ma$ is not positive.

With $f(x)$ thus defined, Laplace found that

$$f(x)\,dx$$

gives, to within infinitesimals of higher order, the probability that the sum of n elements each taken at random from a given range 0 to a $(a>0)$ of uniform distribution, will fall into the interval x to $x+dx$.

Laplace applied the above formula to the historic problem‡ of finding the probability that the inclinations of the orbits of the ten planets besides the earth known at the beginning of the year 1801 do not constitute a random distribution.

When Laplace proceeded to apply his theory to the orbits of all comets§ known at the end of the year 1811, the number n in formula (1) was so large as to render the formula impracticable for numerical computation. Laplace|| used the form

$$(2)\quad \frac{(n+r\sqrt{n})^{n-1} - \binom{n}{1}(n+r\sqrt{n}-2)^{n-1} + \binom{n}{2}(n+r\sqrt{n}-4)^{n-1} - \cdots}{2^n(n-1)!},$$

* Presented to the Society, April 13, 1923.

† See Troisième Edition, 1820, pp. 257–263; cf. E. Czuber, *Wahrscheinlichkeitsrechnung*, vol. I, 1914, p. 66.

‡ Loc. cit., p. 261.

§ Loc. cit., p. 262–3.

|| *Théorie Analytique*, p. 173, Troisième Edition.

where $(n+r\sqrt{n})/2 = x/a$, in place of $\frac{1}{2}f(x)$. He used*

$$\sqrt{\frac{3}{2n\pi}}\,e^{-3r^2/2}\left[1-\frac{3}{20n}(1-6r^2+3r^4)+\cdots\right] \tag{3}$$

as an approximation to (2) and the integral of (3) as an approximation to the corresponding integral of (2).

It was recognized by Cauchy that the methods used by Laplace in obtaining the approximations were lacking in rigor. In fact, Cauchy† put the results on a much more secure basis in a memoir published in 1841. Todhunter‡ described this memoir as "very laborious and difficult, so that this portion of the théorie des probabiltés remains in an unsatisfactory state."

It is the main object of the present paper to put this portion of the theory of probability into a more satisfactory state by developing an approximation to $f(x)$ by means of a Gram-Charlier series. It results that the approximation used by Laplace is given by the first two non-vanishing terms of this series, and that closer and closer approximations in the sense of the theory of least squares are obtained by the use of each additional term of the series. The approximation obtained by retaining three non-vanishing terms of the Gram-Charlier series agrees exactly with the corresponding terms of the Cauchy series, but the derivation of Cauchy does not seem to imply that his approximation is best in the sense of a least squares criterion.

The main difficulty and much of the interest in our method of finding the approximation consists in obtaining remarkably simple expressions for the moments of area under the theoretical frequency curve. Certain auxiliary theorems proved in this connection seem to be new and to be of interest in combinatory analysis.

2. **Symmetry of the curve $y = f(x)$ about the line $x = na/2$.** It will be convenient later in this paper (p. 204) to use the property that the curve $y = f(x)$ is symmetrical about the line $x = na/2$. To prove the symmetry in question, we may first assume that the probability that the sum of the distances of n random points on the line AB from the point A will fall into an assigned interval x_1 to $x_1 + dx$

A ——————— B

0 $\qquad\qquad$ a

* Loc. cit., p. 173.

† *Mémoire sur divers formules relatives à la théorie des intégrales définies et sur la conversion des différences finies des puissances en intégrales de cette espèce*, Journal de L'Ecole Polytechnique, Cahier 28, Tome 17 (1841), pp. 147–248.

‡ Todhunter, *History of Probability*, 1865, p. 527.

is the same as the probability that the sum of the distances of B from the n points will fall into the same interval. Next, it is obvious that a sum of distances of such n points from A falls into the interval x_1 to x_1+dx when and only when the corresponding sum of distances from the points to B falls into the interval $na-x_1-dx$ to $na-x_1$. Hence, the probability that a sum of distances of the n points from A will fall into the interval x_1 to x_1+dx is equal to the probability that the sum will fall into the interval $na-x_1-dx$ to $na-x_1$, and the symmetry of the probability curve $y=f(x)$ about $x=na/2$ is established.

3. **On the convergence of the Gram-Charlier series representing $f(x)$.** The function $y=f(x)$ $(n \geqq 2)$ may obviously be regarded as the sum of a set of rational integral functions with junction points at $x=a, 2a, \cdots, (n-1)a$ and with end points at $x=0$ and $x=na$. Hence, $y=f(x)$ and its successive derivatives are continuous from $x=0$ to $x=na$ unless it be at junction points. It is easily shown* that $y=f(x)=0$ when $x \geqq na$, and we shall find it useful for our purposes to use this extended interval of definition of the function. Furthermore, we shall find it convenient to extend the interval of $y=f(x)$ from $x=0$ to $-\infty$ by making $y=f(x)=0$ throughout this extension. Then the end points $x=0$ and $x=na$ of the original interval 0 to na become junction points.

It follows that $f(x)$ is continuous at any junction point $x=pa$ because the additional term of (1). added for the interval $pa<x<(p+1)a$, has the limit zero as $x \to pa$.

The first derivative of (1),

$$
\begin{aligned}
\frac{dy}{dx} = \frac{1}{a^n(n-2)!}\Big\{x^{n-2} - \binom{n}{1}(x-a)^{n-2} + \binom{n}{2}(x-2a)^{n-2} - \cdots \\
+ (-1)^n \binom{n}{n}(x-na)^{n-2}\Big\},
\end{aligned} \tag{4}
$$

is likewise continuous at the points $x=0, 2a, \cdots, na$, except when $n=2$. When $n=2$ and $x=a$, we have

$$\frac{dy}{dx} = \frac{1}{a^2} \quad \text{or} \quad -\frac{1}{a^2}$$

according as $x \to a$ from below or above a.

* Cf. Chrystal's *Algebra*, Part 2, 1889, p. 210.

The second derivative of (1),

$$
(5)\qquad \frac{d^2y}{dx^2} = \frac{1}{a^n(n-3)!}\left\{x^{n-3} - \binom{n}{1}(x-a)^{n-3} + \binom{n}{2}(x-2a)^{n-3} - \cdots \right.
$$
$$
\left. + (-1)^n \binom{n}{n}(x-na)^{n-3}\right\},
$$

is clearly continuous when $n > 3$. It is continuous when $n = 3$ except at $x = 0$, a, $2a$, and $3a$.

For example, when $n = 3$ and $x = 2a$, we have

$$
\frac{d^2y}{dx^2} = -\frac{2}{a^3} \text{ or } +\frac{1}{a^3}
$$

according as $x \to 2a$ from below or above $2a$.

The examination of the function for continuity could be extended in an obvious manner to higher derivatives, but we shall be interested mainly in the continuity of $f(x)$ and its first and second derivatives.

The theory* of the Gram-Charlier representation of an arbitrary frequency function may be based on the theorem† of W. Myller-Lebedeff proved by the use of integral equations, that any function $g(z)$ which together with its first and second derivatives is finite and continuous from $-\infty$ to $+\infty$ and for which

$$
\lim_{z=\pm\infty} z^3 g(z) = 0
$$

can be represented by an absolutely uniformly convergent infinite series of the form

$$
(6)\qquad c_0\,\Phi_0(z) + c_1\,\Phi_1(z) + c_2\,\Phi_2(z) + \cdots + c_i\,\Phi_i(z) + \cdots,
$$

where $\Phi_i(z)$ is the product of the Gaussian probability function

$$
\Phi_0(z) = \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}
$$

and the ith order polynomial of Hermite $H_i(z)$.

---

* J. P. Gram, Journal für Mathematik, vol. 94 (1883), pp. 41–73; also dissertation, Copenhagen, 1879.

C. L. V. Charlier, *Über die Darstellung willkürlicher Funktionen,* Arkiv for Matematik, Astronomi och Fysik, vol. 2, No. 20 (1905–1906), pp. 1–33. *Vorlesungen über die Grundzüge der mathematischen Statistik,* 1920, pp. 67–78.

Cf. H. Bruns, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre,* 1906, pp. 115–125.

Cf. Mises, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 21 (1912), pp. 9–20.

† W. Myller-Lebedeff, Mathematische Annalen, vol. 64 (1907), p. 400.

We have shown that the conditions of continuity are satisfied when $n > 3$. Obviously the condition

$$\lim_{z=+\infty} x^3 f(x) = 0$$

is also satisfied.

In this representation by (6) the Φ-functions and the H-functions form a biorthogonal system. Thus

$$\int_{-\infty}^{+\infty} \Phi_m(z)\, H_n(z)\, dz = 0 \quad \text{if} \quad m \neq n.$$

The representation of $f(x)$ by such a convergent infinite series naturally suggests that the first few terms of (6) might give a valuable approximation, but the representation is particularly appropriate because it can be shown to be best in the sense of the theory of least squares.

4. **Change of origin and unit.** It is well known* that by both choosing the origin of coördinates so that the x-coördinate of the centroid of area bounded by the graph of the function to be represented and the x-axis is equal to zero, and choosing the standard deviation (radius of gyration), σ, as the unit of measurement, we have

$$c_1 = 0 \quad \text{and} \quad c_2 = 0.$$

We accordingly make the transformation

$$\frac{x - \frac{na}{2}}{\sigma} = z$$

and let

(7) $$y = f(x) = g(z).$$

Moreover, we shall see presently that any coefficient c_i, when i is odd, vanishes because of the symmetry of $y = g(z)$ about $z = 0$ which was proved in § 2. Thus

(8) $$c_i = \frac{(-1)^i}{i!} \int_{-\infty}^{+\infty} g(z) H_i(z)\, dz,$$

* H. Bruns, loc. cit., pp. 118–119.
C. V. L. Charlier, loc. cit., pp. 62–67.

where

$$(9)\qquad H_i(z) = z^i - \frac{i(i-1)}{2} z^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2\cdot 4} z^{i-4} - \cdots$$

is the Hermite polynomial of order i.

The determination of c_i from (8) would, when i is odd, clearly involve merely the sum of a set of odd order moments of area about an axis of symmetry so that each moment is zero. Hence, $c_i = 0$ when i is odd.

We may then write in place of (6)

$$(10)\qquad g(z) = c_0 \Phi_0(z) + c_4 \Phi_4(z) + c_6 \Phi_6(z) + \cdots.$$

Let us examine the finite series

$$(11)\qquad U(z) = c_0 \Phi_0(z) + c_4 \Phi_4(z) + \cdots + c_{2l} \Phi_{2l}(z)$$

as an approximate representation of $g(z)$.

When the coefficients $c_0, c_2, \cdots, c_{2l}$ in (11) are given the values in (8) obtained by the use of the biorthogonal property of the Φ and H-functions, it is known that we have the best approximation of $U(z)$ to $g(z)$ in the sense that a certain least square criterion*

$$(12)\qquad I = \int_{-\infty}^{+\infty} \frac{1}{\Phi_0(z)} [g(z) - U(z)]^2 dz$$

is a minimum.

Moreover, it is known from a general theory of Gram that the approximation of $U(z)$ to $g(z)$ becomes closer† and closer with each increase of l in (11).

In the use of the least squares criterion (12), a question naturally arises as to the propriety of weighting squares of deviations

$$\delta^2 = [g(z) - U(z)]^2$$

with the reciprocal

$$\frac{1}{\Phi_0(z)} = \sqrt{2\pi}\, e^{z^2/2}$$

of the Gaussian probability function. Gram uses this weighting without commenting on its propriety so far as I have been able to find. One fairly

* Gram, loc. cit., p. 54; cf. Mises, loc. cit., p. 20.

† Loc. cit., pp. 47–55.

obvious point in support of such weighting is its algebraic convenience. Another point suggestive of the given weighting is found in the fact that the reciprocal of $\Phi_0(z)$ as a weight tends to weaken* the effect of the square of $\Phi_0(z)$ which occurs as a factor in $[g(z)-U(z)]^2$. Furthermore, it seems natural to have regard for algebraic convenience in this connection because practical statisticians have pointed out the fact that different sets of weights frequently lead to almost identical results. While a sort of rationale thus underlies the scheme of weighting, there remains an element of arbitrariness about it in the sense that it is not necessarily better than some other weighting which might be proposed.

5. **Coefficient of the Φ-series expressed in moments.** We shall give special attention to the approximation by the use of the first three terms of (11) because we obtain the approximation which Laplace used from the first two terms, and a closer approximation in the sense of the theory of least squares by retaining three terms.

The coefficients c_0, c_4, c_6 are now easily expressed in terms of the area $A=1$ under the curve $y=f(x)$ and the moments of this area about the axis $x=na/2$ through the centroid. Let μ_q be the qth moment of area about this axis. Then

$$\mu_q=\int_0^{na}\left(x-\frac{na}{2}\right)^q f(x)\,dx=\sigma^{q+1}\int_{-na/2\sigma}^{na/2\sigma} z^q g(z)\,dz, \tag{13}$$

from the definition of $g(z)$ in (7). Then by means of (8) and (13), we obtain

$$c_0=\frac{1}{\sigma}, \tag{14}$$

$$c_4=\frac{1}{4!\,\sigma}\left[\frac{\mu_4}{\sigma^4}-3\right], \tag{15}$$

$$c_6=\frac{1}{6!\,\sigma}\left[\frac{\mu_6}{\sigma^6}-15\frac{\mu_4}{\sigma^4}+30\right]. \tag{16}$$

By recalling that $\sigma=\sqrt{\mu_2}$, we may say that the difficulties of our problem have now been reduced to that of expressing the moments μ_2, μ_4, and μ_6 in simple form. In this process we shall derive some remarkably simple results from somewhat complicated combinatory forms.

6. **Area and moments of area under $y=f(x)$.** From the definition of $f(x)$ it is fairly obvious that the area, A, is 1, but we present a veri-

* Suggested by Professor E. L. Dodd.

fication of this fact because we lead up in this way to the method by which we have found the moments of area. We should keep especially in mind in this connection that each parenthesis $(x-pa)$ of

$$(17)\quad y=\frac{1}{a^n(n-1)!}\left[x^{n-1}-\binom{n}{1}(x-a)^{n-1}+\binom{n}{2}(x-2a)^{n-1}-\cdots\right.$$
$$\left.+(-1)^n\binom{n}{n}(x-na)^{n-1}\right]$$

is assigned the value zero when $x-pa$ is not positive.

The area would be found by integrating the first term of the right hand member of (17) from 0 to a, the first two terms from a to $2a$, the first three terms from $2a$ to $3a$ and so on. But this process is equivalent to integrating the first term from 0 to na, the second from a to na, the third from $2a$ to na and so on to all the terms. Thus,

$$a^n(n-1)!A=\int_0^{na}x^{n-1}dx-\binom{n}{1}\int_a^{na}(x-a)^{n-1}dx+\binom{n}{2}\int_{2a}^{na}(x-2a)^{n-1}dx-\cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}\int_{(n-1)a}^{na}[x-(n-1)a]^{n-1}dx$$
$$(18)\qquad =\frac{a^n}{n}\left[n^n-\binom{n}{1}(n-1)^n+\binom{n}{2}(n-2)^n-\cdots+(-1)^{n-1}\binom{n}{n-1}\right]$$
$$(19)\qquad =a^n(n-1)!.^*$$

Hence,

$$(20)\qquad A=1.$$

Moments of odd order. The moments of odd order of the area about its centroid vertical $x=na/2$ are each equal to zero, because of the symmetry about this axis. Hence we have

$$\mu_1=\mu_3=\mu_5=\cdots=0.$$

Moments about the axis $x=na$. We shall first find moments about the axis $x=na$, then after considerable simplification, we shall express the moments about the centroidal axis $x=na/2$ in terms of the moments about $x=na$. We let μ'_q represent the qth moment of the area about the axis $x=na$.

* Czuber, *Wahrscheinlichkeitsrechnung*, vol. I, third edition, 1914, p. 66.

(*a*) *Second moments.* In this case we have

$$a^n(n-1)!\mu_2' = \int_0^{na} x^{n-1}(x-na)^2dx - \binom{n}{1}\int_a^{na}(x-a)^{n-1}(x-na)^2dx + \cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}\int_{na-a}^{na}(x-na+a)^{n-1}(x-na)^2dx$$
$$= \int_0^{na} x^{n-1}(x-na)^2dx - \binom{n}{1}\int_a^{na}(x-a)^{n-1}[x-a-(n-1)a]^2dx + \cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}\int_{na-a}^{na}(x-na+a)^{n-1}(x-na+a-a)^2dx$$
$$= \left(\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}\right)a^{n+2}\left[n^{n+2} - \binom{n}{1}(n-1)^{n+2} + \binom{n}{2}(n-2)^{n+2} - \cdots\right.$$
$$\left.+(-1)^{n-1}\binom{n}{n-1}\right]$$
$$= \frac{2a^{n+2}}{n(n+1)(n+2)}\left[n^{n+2} - \binom{n}{1}(n-1)^{n+2} + \binom{n}{2}(n-2)^{n+2} - \cdots\right.$$
$$\left.+(-1)^{n-1}\binom{n}{n-1}\right],$$

$$(21)\qquad \mu_2' = \frac{2a^2}{(n+2)!}\left[n^{n+2} - \binom{n}{1}(n-1)^{n+2} + \binom{n}{2}(n-2)^{n+2} - \cdots\right.$$
$$\left.+(-1)^{n-1}\binom{n}{n-1}\right],$$

where the terms $(n-p)$ are assigned the value zero whenever $n-p$ is not positive.

(*b*) *Fourth moments.* In this case,

$$a^n(n-1)!\mu_4' = \int_0^{na} x^{n-1}(x-na)^4dx - \binom{n}{1}\int_a^{na}(x-a)^{n-1}(x-na)^4dx + \cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}\int_{na-a}^{na}(x-na+a)^{n-1}(x-na)^4dx$$
$$= \int_0^{na} x^{n-1}(x-na)^4dx - \binom{n}{1}\int_a^{na}(x-a)^{n-1}[x-a-na+a]^4 + \cdots$$
$$+(-1)^{n-1}\binom{n}{n-1}\int_{na-a}^{na}(x-na+a)^{n-1}(x-na+a-a)^4dx$$

$$= \left(\frac{1}{n+4} - \frac{4}{n+3} + \frac{6}{n+2} - \frac{4}{n+1} + \frac{1}{n}\right) a^{n+4} \Big[n^{n+4} - \binom{n}{1} (n-1)^{n+4}$$

$$+ \binom{n}{2} (n-2)^{n+4} - \cdots + (-1)^{n-1} \binom{n}{n-1} \Big]$$

$$= \frac{a^{n+4}\, 4!}{n(n+1)(n+2)(n+3)(n+4)} \Big[n^{n+4} - \binom{n}{1} (n-1)^{n+4}$$

$$+ \binom{n}{2} (n-2)^{n+4} - \cdots + (-1)^{n-1} \binom{n}{n-1} \Big]$$

$$\mu_4' = \frac{a^4\, 4!}{(n+4)!} \Big[n^{n+4} - \binom{n}{1} (n-1)^{n+4} + \binom{n}{2} (n-2)^{n+4} - \cdots \tag{22}$$

$$+ (-1)^{n-1} \binom{n}{n-1} \Big].$$

(c) *General formula for moments of order m (m any positive integer).* By a process similar to that employed for second and fourth moments, we find the mth moment about $x = na$ to be

$$\mu_m' = (-1)^m \frac{a^m\, m!}{(n+m)!} \Big[n^{n+m} - \binom{n}{1} (n-1)^{n+m} + \binom{n}{2} (n-2)^{n+m} - \cdots \tag{23}$$

$$+ (-1)^{n-1} \binom{n}{n-1} \Big],$$

where m is any positive integer.

7. **On the reduction of certain combinatory forms.** The moments of area about the axis $x = na$ obtained in § 6 involve combinatory expressions of the form

$$F(n, t) = \frac{t!}{(n+t)!} \Big[n^{n+t} - \binom{n}{1} (n-1)^{n+t} + \binom{n}{2} (n-2)^{n+t} - \cdots \tag{24}$$

$$+ (-1)^{n-1} \binom{n}{n-1} \Big],$$

where t is an integer $\geqq 1$.

Although it is well known that $F(n, 0) = 1$, and that the expression in brackets in the right hand member of (24) is equal* to zero when t is a negative integer $(n + t > 0)$, no simple values seem to have been found for $F(n, t)$ when $t \geqq 1$, so far as the writer has been able to learn.

In what follows, we obtain simple polynomials in n for $F(n, t)$ when $t = 1, 2, 3, \cdots, 6$, and find (§ 9) that the results lead to very important simplifications in the first six moments given in § 5. Similar results could

* Picquet, Journal de Mathématiques Spéciales, ser. 3, vol. 2 (1888), pp. 150, 172, 196. E. Netto, Lehrbuch der Combinatorik, 1901, p. 37.

be obtained for somewhat larger values of t if they were needed. In fact the writer has derived the results for $t = 7$ and $t = 8$. The following theorems giving $F(n, t)$ for $t = 1, 2, 3, \cdots, 6$, in the form of polynomials will be proved:

$$F(n, 1) = \frac{n}{2}; \tag{25}$$

$$F(n, 2) = \frac{n(3n+1)}{12}; \tag{26}$$

$$F(n, 3) = \frac{n^2(n+1)}{8}; \tag{27}$$

$$F(n, 4) = n\left(\frac{n^3}{16} + \frac{n^2}{8} + \frac{n}{48} - \frac{1}{120}\right); \tag{28}$$

$$F(n, 5) = \frac{n^2}{8}\left(\frac{n^3}{4} + \frac{5n^2}{6} + \frac{5n}{12} - \frac{1}{6}\right); \tag{29}$$

$$F(n, 6) = \frac{n}{4}\left(\frac{n^5}{16} + \frac{5n^4}{16} + \frac{5n^3}{16} - \frac{13n^2}{144} - \frac{n}{24} + \frac{1}{63}\right). \tag{30}$$

It may be observed from (23) and (24) that the left hand members of (25), (27) and (29) are involved in odd order moments of area about the axis $x = na$.

Thus for the first moment about the axis $x = na$ we obtain from (23)

$$\mu_1' = -\frac{a}{(n+1)!}\left[n^{n+1} - \binom{n}{1}(n-1)^{n+1} + \binom{n}{2}(n-2)^{n+1} - \cdots + (-1)^{n-1}\binom{n}{n-1}\right].$$

Then using (24) with $t = 1$,

$$\mu_1' = -aF(n, 1). \tag{31}$$

But

$$\mu_1' = -\frac{na}{2}, \tag{32}$$

since the first moment coefficient μ_1' is simply the distance from the axis of moments to the centroid.

From (31) and (32), we have

$$F(n, 1) = \frac{n}{2}. \tag{33}$$

Before presenting the proofs of the equalities (26) to (30), it may be of interest to explain what suggested the values given for $F(n, t)$ in each case. The value of $F(n, 2)$ was suggested by experimenting with the special cases $n = 1, 2, \cdots, 6$. The values of $F(n, 3)$ and $F(n, 5)$ were suggested by equating third and fifth order moments as given by (23) to the corresponding values found by transforming the odd order moments about $x = na/2$ to the parallel axis $x = na$. The values of $F(n, 4)$ and $F(n, 6)$ were suggested by finding the fourth and sixth moments about $x = na/2$ of the area under the Gaussian probability curve which would fit $y = f(x)$ best, and equating these expressions for the moments to the values of the corresponding moments μ_4 and μ_6 obtained from μ_4' and μ_6' given in § 6, and then experimenting with corrections until a correctional formula was found which could be proved to hold for any integral value of n.

We shall now prove (26) to (30) by mathematical induction. First, verify the relations for small values of n, say for $n = 1, 2, 3$.

In the proof, we make much use of the relation*

$$(34) \qquad (k+t)F(k, t) = k[tF(k, t-1) + F(k-1, t)],$$

which is easily established by substitution from the definition of $F(n, t)$ given in (24), or by the use of a theorem† of combinatory analysis to which (34) is closely related.

By means of (34) we can now prove that each of the equalities (26) to (30) is true for a positive integer $n = k$ if it is true for $n = k-1$.

(a) First consider formula (26). From (34),

$$\begin{aligned}(k+2)F(k, 2) &= k\{2F(k, 1) + F(k-1, 2)\}\\ &= k\left\{k + \frac{k-1}{12}[3(k-1)+1]\right\},\end{aligned}$$

since from (33), $F(k, 1) = k/2$, and since we are assuming the equality (26) for $n = k-1$. Solving for $F(k, 2)$, we have

$$F(k, 2) = \frac{k(3k+1)}{12},$$

which was to be proved.

* Cf. J. Worpitzky, *Studien über die Bernoullischen und Eulerschen Zahlen*, Journal für Mathematik, vol. 94 (1883), p. 210.

† E. Netto, *Lehrbuch der Combinatorik*, 1901, p. 169.

(*b*) Formula (27). From (34),

$$(k+3)F(k,3) = k[3F(k,2)+F(k-1,3)]$$
$$= k\left[\frac{k(3k+1)}{4}+\frac{k(k-1)^2}{8}\right] = \frac{k^2}{8}(k+1)(k+3),$$

since $F(k,2) = k(3k+1)/12$, and we assume (27) for $n = k-1$. Hence,

$$F(k,3) = \frac{k^2}{8}(k+1),$$

which was to be proved.

(*c*) Formula (28). From (34),

$$(k+4)F(k,4) = k[4F(k,3)+F(k-1,4)]$$
$$= k\left[\frac{k^2(k+1)}{2}+\frac{(k-1)^4}{16}+\frac{(k-1)^3}{8}+\frac{(k-1)^2}{48}-\frac{(k-1)}{120}\right]$$
$$= k(k+4)\left(\frac{k^3}{16}+\frac{k^2}{8}+\frac{k}{48}-\frac{1}{120}\right),$$

since $F(k,3) = k^2(k+1)/8$, and we assume (28) for $n = k-1$. Hence,

$$F(k,4) = k\left(\frac{k^3}{16}+\frac{k^2}{8}+\frac{k}{48}-\frac{1}{120}\right),$$

which was to be proved.

(*d*) Formula (29). From (34),

$$(k+5)F(k,5) = k[5F(k,4)+F(k-1,5)]$$
$$= k\left[\frac{5k^4}{16}+\frac{5k^3}{8}+\frac{5k^2}{48}-\frac{k}{24}+\frac{(k-1)^5}{32}+\frac{5(k-1)^4}{48}\right.$$
$$\left.+\frac{5(k-1)^3}{96}-\frac{(k-1)^2}{48}\right]$$
$$= \frac{k^2}{8}(k+5)\left(\frac{k^3}{4}+\frac{5k^2}{6}+\frac{5k}{12}-\frac{1}{6}\right).$$

Hence,

$$F(k,5) = \frac{k^2}{8}\left(\frac{k^3}{4}+\frac{5k^2}{6}+\frac{5k}{12}-\frac{1}{6}\right),$$

which was to be proved.

(*e*) Formula (30). From (34),

$$\begin{aligned}(k+6)F(k,6) &= k[6F(k,5)+F(k-1,6)]\\ &= k\left[\frac{6k^5}{32}+\frac{5k^4}{8}+\frac{5k^3}{16}-\frac{k^2}{8}+\frac{(k-1)^6}{64}+\frac{5(k-1)^5}{64}\right.\\ &\qquad\left.+\frac{5(k-1)^4}{64}-\frac{13(k-1)^3}{576}-\frac{(k-1)^2}{96}+\frac{k-1}{252}\right]\\ &= \frac{k}{4}(k+6)\left(\frac{k^5}{16}+\frac{5k^4}{16}+\frac{5k^3}{16}-\frac{13k^2}{144}-\frac{k}{24}+\frac{1}{63}\right).\end{aligned}$$

Hence,

$$F(k,6) = \frac{k}{4}\left(\frac{k^5}{16}+\frac{5k^4}{16}+\frac{5k^3}{16}-\frac{13k^2}{144}-\frac{k}{24}+\frac{1}{63}\right),$$

which was to be proved.

8. **Simplification of moments.** By the use of auxiliary theorems proved in § 7, we are now able to write the moments of area about the axis $x = na$ given in § 6 in greatly simplified forms. Thus, by substitution from (26), (28) and (33) in (23) we obtain

$$\mu_2' = \frac{n}{12}(3n+1)a^2, \tag{35}$$

$$\mu_4' = \frac{n}{8}\left(\frac{n^3}{2}+n^2+\frac{n}{6}-\frac{1}{15}\right)a^4, \tag{36}$$

$$\mu_6' = \frac{n}{4}\left(\frac{n^5}{16}+\frac{5n^4}{16}+\frac{5n^3}{16}-\frac{13n^2}{144}-\frac{n}{24}+\frac{1}{63}\right)a^6. \tag{37}$$

9. **Even order moments about the axis $x = na/2$ through the centroid of area.** The moments μ_2, μ_4, μ_6 about the parallel axis through the centroid are given in terms of μ_2', μ_4', μ_6', and the distance $d = na/2$ between the parallel axes, by the well known relations

$$\mu_2 = \mu_2' - d^2, \tag{38}$$

$$\mu_4 = \mu_4' + 4\mu_3 d - 6\mu_2 d^2 + 4\mu_1 d^3 - d^4, \tag{39}$$

$$\mu_6 = \mu_6' + 6\mu_5 d - 15\mu_4 d^2 + 20\mu_3 d^3 - 15\mu_2 d^4 + 6\mu_1 d^5 - d^6. \tag{40}$$

Substituting in (38), (39) and (40) from (35), (36) and (37), remembering that odd order moments μ_1, μ_3, μ_5 are each zero, we have

(41) $$\mu_2 = \frac{a^2 n}{12},$$

(42) $$\mu_4 = \frac{a^4 n}{24}\left(\frac{n}{2} - \frac{1}{5}\right),$$

(43) $$\mu_6 = \frac{a^6 n}{12}\left(\frac{5n^2}{48} - \frac{n}{8} + \frac{1}{21}\right).$$

10. **The coefficients of the $\Phi(z)$-series.** By substitution for the moments μ_2, μ_4, and μ_6 in (15) and (16) their simple values given in (41), (42) and (43), we have

(44) $$c_4 = -\frac{1}{20n\sigma},$$

and

(45) $$c_6 = +\frac{1}{105n^2\sigma},$$

which seem to be remarkably simple results.

Hence, we have, from (10), (14), (44) and (45), for the approximate representation of our frequency function

(46) $$y = \frac{1}{\sigma}\left[\varphi_0(z) - \frac{1}{20n}\varphi_4(z) + \frac{1}{105n^2}\varphi_6(z)\right],$$

where

$$\varphi_0(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

$$z = \frac{x - \frac{na}{2}}{\sigma},$$

and

$$\sigma^2 = \frac{a^2 n}{12}.$$

The equation (46) may also be written in the form

(47) $$y = \frac{1}{a}\sqrt{\frac{6}{n\pi}}\, e^{-z^2/2}\left[1 - \frac{1}{20n}(z^4 - 6z^2 + 3) + \frac{1}{105n^2}(z^6 - 15z^4 + 45z^2 - 15)\right].$$

The first term of (46) is the Gaussian probability function. The sum of the first two terms divided by 2 is identical with the approximation which Laplace used for formula (2) as may be shown by making the transformations

$$z = \frac{x - \frac{na}{2}}{\sigma}, \qquad x = \frac{n + r\sqrt{n}}{2}, \qquad a = 1.$$

11. **Numerical checks on approximations.** Although it is known (§ 4) that our approximate representation of $y = f(x)$ is improved by the inclusion of each additional term of (11) in the sense of minimizing a certain least squares criterion, we find by numerical computation that the approximation is not in all cases improved at every point by the inclusion of an additional term. In order to emphasize this fact, and to gain an insight into the progress of the approximation from term to term, the writer has computed tables of $f(x)$ and of the corresponding approximations for some simple values of n. The numerical computations are much facilitated by the use of published tables of $\Phi_0(z)$ and of its first six derivatives. A set of such tables to seven decimal places for values of z up to 4 at intervals of 0.01 are published in N. R. Jorgensen,* and a similar set of five-place tables to eight derivatives for z up to 5 are published by James W. Glover.†

In my numerical computations, the ordinates $y = f(x)$ have ordinarily been found at the junction points described in § 3, because it is but natural to expect that the approximations might not be so good at these points as elsewhere. Thus, the values of $f(x)$ and of the first, second, and third approximations, given by one, two and three terms of (46) respectively, have been computed at junction points for $n = 12$. It turns out that at the point $x = 0$, the second approximation is not so near the value of $f(x)$ as the first approximation and that at the three points $x = 4a$, $5a$, and $6a$, the third approximation is not so near the corresponding values of $f(x)$ as the second approximation. This simple case illustrates well the fact that the approximation is not improved at every point by the inclusion of an additional term, although it is improved in the sense of our least squares criterion.

* *Frequensflader og Korrelation*, 1916, pp. 177–193.

† *Tables for Applied Mathematics*, 1923, pp. 391–411.

University of Iowa,
Iowa City, Iowa.

THE GROUP OF MOTIONS OF AN EINSTEIN SPACE*

BY

JOHN EIESLAND

Introduction

The question to what extent the general Einstein space is determined by its group of motions seems to be of interest from a physical as well as a geometric standpoint.

In what follows we have discussed the problem of determining the group of motions in a given Riemannian n-space and its converse (Killing's equations). The assumption is then made that an Einstein space whose linear element is

$$(a) \qquad -ds^2 = \sum_1^3 g_{ik}\, dx_i\, dx_k - \sum_0^3 g_{i0}\, dx_i\, dx_0, \qquad x_0 = t,$$

shall admit the group of "rotations"

$$G_3: \qquad x^i \frac{\partial f}{\partial x_k} - x_k \frac{\partial f}{\partial x_i} \qquad (i, k = 1, 2, 3)$$

and the following theorem is proved:

A necessary and sufficient condition that the space (*a*) *shall be reducible to the form*

$$(b) \qquad -ds^2 = \varphi_2\, dr^2 + \varphi_3 (d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2,$$

φ_1, φ_2 and φ_3 being arbitrary functions of r and t, is that it shall admit the group G_3 as a complete group of motions.†

It may further be required that (*b*) admit a one-parameter group

$$\xi_0 \frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial r},$$

where ξ_0 and ξ_1 are functions of r and t. The necessary and sufficient conditions that this shall be the case are found to be

* Presented to the Society, March 26, 1921.

† This theorem has generally been taken for granted by writers on relativity.

$$\varphi_1 \varphi_3 G_1^0 = \Psi'(\varphi_3) \frac{\partial \varphi_3}{\partial t} \frac{\partial \varphi_3}{\partial r},$$

$$\varphi_1 \varphi_2 \varphi_3 (G_1^1 - G_0^0) = +2\Psi'(\varphi_3)\left[\varphi_1 \frac{\partial \varphi_3}{\partial r} + \varphi_2 \frac{\partial \varphi_3}{\partial t}\right],$$

(Ψ an arbitrary function of φ_3 or a constant). It is then shown that if these conditions are satisfied, the space (b) may be reduced to the static form.

Special forms of static spaces are then considered with special reference to their group properties and the principal curvature of their sub-spaces

$$S_3\colon\ t = 0; \qquad S_3^1\colon\ \varphi = 0; \qquad S_3^2\colon\ \theta = 0.$$

The question of the class of the quadratic form (b) is then taken up, and it is proved that a necessary and sufficient condition that (b) shall be of class 1 is

$$(c) \qquad (02,02)(13,13) = (01,01)(23,23) + (12,02)(13,03).$$

The general space (b) can therefore be immersed in a flat 6-space, and, if (c) is satisfied, in a flat 5-space.

It is also proved that if a general space (a) admits any one of the abelian groups

$$\frac{\partial f}{\partial x_0}; \quad \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_3}; \quad \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_2},$$

as complete group of motions, it is of the fifth, third, and second class respectively. Among these spaces is found Weyl's static and cylindrical space admitting an abelian G_2.

1. **The general differential quadratic form.** Let there be given a general differential quadratic form

$$(1) \qquad ds^2 = \sum_1^n a_{ik}\, dx_i\, dx_k$$

which may be interpreted as the linear element of a curved space S_n of n dimensions. This space is said to admit of a group of rigid motions, if there exists a group of transformations

$$(2) \qquad x_i' = f_i(x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_r) \qquad (i = 1, 2, \cdots, n),$$

which will carry the form (1) into the form

$$ds^2 = \sum_1^n a_{ik}'\, dx_i'\, dx_k'$$

such that the coefficients a'_{ik} are the same functions of x'_i as the a_{ik}'s are of x_i. If the coefficients a_{ik} are perfectly general, no such group exists.

In order that a given form (1) shall admit a group*

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n},$$

the ξ's must satisfy the so-called Killing's equations,

$$(3) \qquad \sum_1^n \xi_\lambda \frac{\partial a_{ik}}{\partial x_\lambda} + \sum_1^n a_{i\lambda} \frac{\partial \xi_\lambda}{\partial x_k} + \sum_1^n a_{k\lambda} \frac{\partial \xi_\lambda}{\partial x_i} = 0 \qquad (i, k = 1, 2, \cdots, n),$$

the integration of which will determine the ξ's as functions of x_i and r constants of integration. The maximum group has $r = n(n+1)/2$ parameters, in which case the space S_n has a constant Riemannian curvature. Bianchi† gives Killing's equations another form.

$$(4) \qquad \frac{\partial \eta_i}{\partial x_k} + \frac{\partial \eta_k}{\partial x_i} = 2 \sum_\lambda^{1\cdots n} \left\{ \begin{matrix} ik \\ \lambda \end{matrix} \right\} \eta_\lambda, \quad \eta_i = \sum_\lambda^{1\cdots n} a_{i\lambda} \xi_\lambda \qquad (i, k = 1, 2, \cdots, n),$$

where $\left\{ \begin{matrix} ik \\ \lambda \end{matrix} \right\}$ are the usual Christoffel symbols. All the second derivatives obtained from these equations can be expressed linearly and homogeneously in terms of the η's and their first derivatives. We thus obtain the system

$$(5) \qquad \frac{\partial^2 \eta_i}{\partial x_k \partial x_l} = \frac{\partial}{\partial x_l} \sum_\lambda^{1\cdots n} \left\{ \begin{matrix} ik \\ \lambda \end{matrix} \right\} \eta_\lambda + \frac{\partial}{\partial x_k} \sum_\lambda^{1\cdots n} \left\{ \begin{matrix} il \\ \lambda \end{matrix} \right\} \eta_\lambda - \frac{\partial}{\partial x_i} \sum_\lambda^{1\cdots n} \left\{ \begin{matrix} kl \\ \lambda \end{matrix} \right\} \eta_\lambda$$
$$(i, k, l = 1, 2, \cdots, n).$$

If the systems (4) and (5) are completely integrable, the group has $r = n(n+1)/2$ parameters. If $r < n(n+1)/2$, the system is not complete. If therefore we form the conditions of integrability, we find new relations between the η's and their first derivatives which must be added to the system (4). Continuing in this way we shall eventually arrive at the complete Lie-Mayer system defining the group.

2. Let us suppose that a space S_n admits at least a one-parameter group G_1. By proper choice of variables this group may always be reduced

---

* It is clear that if (1) is invariant under the ∞^r finite transformations of the group (2) it is also invariant under the corresponding r infinitesimal transformations of the group. For proof of the converse see L. Bianchi, *Lezioni sulla Teoria dei Gruppi Continui di Transformazioni*, Pisa, 1918, pp. 493–495.

† L. Bianchi, loc. cit., pp. 502–503.

to the form $\partial f/\partial x_1$. If therefore we put $\xi_1 = 1$, $\xi_2 = \xi_3 = \cdots = \xi_n = 0$ in Killing's equations (3), we find

$$\frac{\partial a_{ik}}{\partial x_1} = 0$$

which means that the coefficients a_{ik} do not contain x_1. Hence, *if a space S_n admits a one-parameter group of motions, its linear element can always be put in the form*

$$ds^2 = \sum_0^3 a_{ik}\, dx_i\, dx_k, \tag{6}$$

*where the coefficients a_{ik} do not contain x_1.**

Suppose further that the group G_1 is such that the infinitesimal motion at every point of S_n has a constant amplitude. A motion of this kind corresponds to a translation in ordinary euclidean space (Schiebung). Since we have $\delta x_i = \xi_i \delta t$, the condition to be satisfied, in addition to those of equations (3), is

$$\frac{\delta s^2}{\delta t^2} = \sum_0^3 a_{ik}\, \xi_i\, \xi_k = \text{const.}; \tag{7}$$

the ξ's are therefore the constants of direction at any point in S_n. If we reduce ds^2 to the form (6) and apply (7) we find $a_{11} = \text{const.}$ But the condition $a_{11} = \text{const.}$ is the condition that the line x_1 shall be a geodesic in S_n.† We have therefore the

THEOREM I. *An infinitesimal motion is a translation if, and only if, the trajectories of the group G_1 generated by it are geodesic lines in S_n.*

Any finite translation carries all the points of space the same geodesic distance from their original positions.

We shall state the following proposition,‡ the proof of which we shall omit:

If the space S_n admits a translation, any spread formed by ∞^1 trajectories of the motion is of zero curvature.

3. **The space of a four-dimensional metric field.** After these preliminaries which are largely restatements of well known theorems we shall proceed to study the four-dimensional metric field of Einstein's relativity theory, with a special view to its group-theoretical properties.

* The converse is also true: If the linear element of S_n can be put in the form (6), the space admits at least a one-parameter group of motions.

† L. Bianchi, loc. cit., p. 500.

‡ Loc. cit., p. 500.

Consider the quadratic form

$$(8) \qquad ds^2 = \sum_0^3 g_{ik}\, dx_i\, dx_k,$$

which in Einstein's relativity theory contains the metrical relations of time and physical space. Let $x_0 = t$, t being interpreted as time, and let x_1, x_2, x_3 be the coördinates of a space such that its linear element

$$ds_0^2 = -(ds^2)_{dx_0=0}.$$

We may therefore put

$$(9) \qquad ds_0^2 = -\sum_1^3 g_{ik}\, dx_i\, dx_k = \sum_1^3 a_{ik}\, dx_i\, dx_k,$$

and we shall assume moreover that this form is positive and definite. The general quadratic form (9) may therefore be written

$$(10) \qquad ds^2 = g_{00}\, dt^2 + \sum_0^3 g_{0i}\, dx_i\, dt - \sum_1^3 a_{ik}\, dx_i\, dx_k$$

which is indefinite, the index of inertia being 3. g_{00} may be interpreted as a velocity; for, if t only varies, we have $ds^2/dt^2 = g_{00} = V^2$ so that $\sqrt{g_{00}} = V$ has the dimension of velocity.

Let us assume that the coefficients g_{00}, g_{i0} and a_{ik} do not contain t. By Theorem I this means that (10) admits at least a one-parameter group of motions, namely

$$(11) \qquad Uf = \frac{\partial f}{\partial t},$$

the invariant spreads of which are the 3-spreads $t =$ const. A space of this kind we shall call with Levi-Civita a *stationary space*, so that we have the

THEOREM II. *A necessary and sufficient condition that a general Einstein space* (10) *shall be stationary is that it shall admit the group* (11). *This motion is a "translation" if, and only if,* g_{00} *is a constant* (*Theorem* I).

The path-curves of the transformation (11) are not in general geodesics in S_4. Only when g_{00} is a constant will this be the case, and (10) may be reduced to the geodesic form

$$(12) \qquad ds^2 = c^2 dt^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k,$$

in which the coefficients g_{i0} are absent and the new coefficients a_{ik} do not contain t as before. Since ∞^1 paths-curves of the "translation" will

form a spread of zero curvature, the space (12) may be described as "cylindrical".*

We shall, however, assume that the space (10) is general, the coefficients g_{ik} being functions of x_i, $i = 0, 1, 2, 3$. By means of the transformation

$$x_0' = x_0, \quad x_i' = x_i'(x_0, x_1, x_2, x_3) \qquad (i = 1, 2, 3)$$

we may remove the coefficients g_{i0} in (10); in fact, it will be necessary and sufficient that the functions x_i' shall be solutions of the differential equation

$$\nabla(x_0, \theta) = \sum_0^3 g^{i0} \frac{\partial \theta}{\partial x_i} = 0,$$

$\nabla(x_0, x_i') = 0$ being the conditions that the space x_1', x_2', x_3' shall be orthogonal to the coördinate line x_0'. The space (10) has now the form

$$ds^2 = g_{00}\, dx_0^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k. \tag{13}$$

Let us suppose that this space admits a group G and let the general nature of this group be left arbitrary for the time being, except that it does not operate on $x_0 = t$, i. e., it is a group of the sub-space $x_0 =$ const. We write then

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3}. \tag{14}$$

The equations (3) are

$$\begin{gathered}
\sum_1^3 \xi_\lambda \frac{\partial g_{00}}{\partial x_\lambda} + 2\sum_1^3 g_{0\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 \xi_\lambda \frac{\partial a_{11}}{\partial x_\lambda} + 2\sum_1^3 a_{1\lambda} \frac{\partial \xi_\lambda}{\partial x_1} = 0,\\
\sum_1^3 \xi_\lambda \frac{\partial a_{22}}{\partial x_\lambda} + 2\sum_1^3 a_{2\lambda} \frac{\partial \xi_\lambda}{\partial x_2} = 0, \qquad \sum_1^3 \xi_\lambda \frac{\partial a_{33}}{\partial x_\lambda} + 2\sum_1^3 a_{3\lambda} \frac{\partial \xi_\lambda}{\partial x_3} = 0,\\
\sum_1^3 \xi_\lambda \frac{\partial a_{ik}}{\partial x_\lambda} + \sum_1^3 a_{i\lambda} \frac{\partial \xi_\lambda}{\partial x_\lambda} + \sum_1^3 a_{k\lambda} \frac{\partial \xi_\lambda}{\partial x_\lambda} = 0,\\
\sum_1^3 a_{1\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 a_{2\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 a_{3\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0.
\end{gathered} \tag{15}$$

* The term "static" instead of "stationary" has been used by G. D. Birkhoff in a recent publication, *Relativity and Modern Physics* (Cambridge, Harvard University Press, 1923). If we consider the hydrodynamic analogy, it would seem that the term "stationary" is a better term. We do not speak of a "static" motion in hydrodynamics, when a stationary or permanent motion is meant. The term "static" field is used by T. Levi-Civita to denote a stationary field in which the coefficients g_{i0} are absent. See T. Levi-Civita, *La Teoria di Einstein e il Principio di Fermat*, Il Nuovo Cimento, ser. 6, vol. 16 (1918), pp. 105–114.

Since the determinant $|a_{ik}|$ cannot vanish, the last three equations show that the ξ's are independent of x_0. The first of equations (15) becomes

$$\sum_1^3 \xi_\lambda \frac{\partial g_{00}}{\partial x_\lambda} = 0,$$

which means that g_{00} is of the form $g_{00}(\varphi, x_0)$ where φ is an invariant of the group G, or else a function of x_0 alone, in which case g_{00} may be reduced to a constant. In the first case, since G does not involve x_0, g_{00} is itself an invariant of the group.

(*a*) g_{00} *an invariant of* G. G must be an intransitive group considered as belonging to S_4. But since G does not contain x_0, nor operate on x_0, it must be a group of motions in S_3; this is also clear when we consider that the remaining equations in the system (15) are Killing's equations corresponding to the space S_3. It should be noted that this does not prevent G from being a subgroup of a transitive group $\bar{G}$ of motions in S_3, but G will not belong to S_4 unless g_{00} is a function of x_0 alone, or a constant.

(*b*) $g_{00} = \text{const.}$ In this case G may be any group of motions in S_3, transitive or intransitive; it may even be the maximum group G_6 in which case S_3 is a space of constant positive or zero curvature. If G is a transitive group in S_3, it can belong to S_4 if, and only if, g_{00} is a function of x_0 alone or a constant. We shall state these results in the following

THEOREM III. *If the space whose linear element is*

$$ds^2 = g_{00}\, dx_0^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k$$

admits a group of the form

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3},$$

the ξ's are independent of x_0, and the group belongs also to the subspace S_3. g_{00} is either an invariant of the group, or a function of x_0 alone. In the first case, the group is intransitive. In the second case, g_{00} may by a transformation be reduced to a constant, and the group is either transitive or intransitive. A transitive group in S_3 belongs to S_4 if, and only if, g_{00} is a constant.

4. **The group of "rotations" in S_3.*** We shall suppose that S_4 admits the intransitive group of "rotations" about the origin in S_3, viz.

$$(16) \qquad x_1 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1}, \quad x_2 \frac{\partial f}{\partial x_3} - x_3 \frac{\partial f}{\partial x_2}, \quad x_3 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_3}.$$

* By a "group of rotations" we mean here a 3-parameter group in the variables x_1, x_2, x_3 having the invariant $x_1^2 + x_2^2 + x_3^2$; x_1, x_2, x_3 are not cartesian coördinates.

The sub-space S_3 is then said to be *centro-symmetric*. If in (15) we introduce in succession the following values of the ξ's,

$$\begin{array}{lll} \xi_1 = -x_2, & \xi_2 = x_1, & \xi_3 = 0, \\ \xi_1 = 0, & \xi_2 = -x_3, & \xi_3 = x_2, \\ \xi_1 = x_3, & \xi_2 = 0, & \xi_3 = -x_1, \end{array}$$

we obtain a system of equations for determining the quantities g_{00} and a_{ik}:

$$(a)\qquad x_1\frac{\partial g_{00}}{\partial x_2}-x_2\frac{\partial g_{00}}{\partial x_1}=0,\quad x_2\frac{\partial g_{00}}{\partial x_3}-x_3\frac{\partial g_{00}}{\partial x_2}=0,\quad x_3\frac{\partial g_{00}}{\partial x_1}-x_1\frac{\partial g_{00}}{\partial x_3}=0;$$

$$(17)\ (b)\qquad \begin{aligned}
&x_1\frac{\partial a_{11}}{\partial x_2}-x_2\frac{\partial a_{11}}{\partial x_1}+2a_{12}=0,\quad x_2\frac{\partial a_{11}}{\partial x_3}-x_3\frac{\partial a_{11}}{\partial x_2}=0,\\
&x_3\frac{\partial a_{11}}{\partial x_1}-x_1\frac{\partial a_{11}}{\partial x_3}-2a_{13}=0,\\
&x_1\frac{\partial a_{22}}{\partial x_2}-x_2\frac{\partial a_{22}}{\partial x_1}-2a_{12}=0,\quad x_2\frac{\partial a_{22}}{\partial x_3}-x_3\frac{\partial a_{22}}{\partial x_2}+2a_{23}=0,\\
&x_3\frac{\partial a_{22}}{\partial x_1}-x_1\frac{\partial a_{22}}{\partial x_3}=0,\\
&x_1\frac{\partial a_{33}}{\partial x_2}-x_2\frac{\partial a_{33}}{\partial x_1}=0,\quad x_2\frac{\partial a_{33}}{\partial x_3}-x_3\frac{\partial a_{33}}{\partial x_2}-2a_{23}=0,\\
&x_3\frac{\partial a_{33}}{\partial x_1}-x_1\frac{\partial a_{33}}{\partial x_3}+2a_{13}=0,\\
&x_1\frac{\partial a_{12}}{\partial x_2}-x_2\frac{\partial a_{12}}{\partial x_1}-a_{11}+a_{22}=0,\quad x_2\frac{\partial a_{12}}{\partial x_3}-x_3\frac{\partial a_{12}}{\partial x_2}+a_{13}=0,\\
&x_3\frac{\partial a_{12}}{\partial x_1}-x_1\frac{\partial a_{12}}{\partial x_3}-a_{23}=0,\\
&x_1\frac{\partial a_{13}}{\partial x_2}-x_2\frac{\partial a_{13}}{\partial x_1}+a_{23}=0,\quad x_2\frac{\partial a_{13}}{\partial x_3}-x_3\frac{\partial a_{13}}{\partial x_2}+a_{12}=0,\\
&x_3\frac{\partial a_{13}}{\partial x_1}-x_1\frac{\partial a_{13}}{\partial x_3}-a_{33}+a_{11}=0,\\
&x_1\frac{\partial a_{23}}{\partial x_2}-x_2\frac{\partial a_{23}}{\partial x_1}-a_{13}=0,\quad x_2\frac{\partial a_{23}}{\partial x_3}-x_3\frac{\partial a_{23}}{\partial x_2}+a_{22}-a_{33}=0.\\
&x_3\frac{\partial a_{23}}{\partial x_1}-x_1\frac{\partial a_{23}}{\partial x_3}+a_{12}=0.
\end{aligned}$$

If we suppose that g_{00} is not a function of x_0 alone, the equations (a) express the fact that g_{00} is an invariant of the group G_3, so that we may put

$$g_{00} = \varphi_1(\sqrt{x_1^2+x_2^2+x_3^2},\ x_0).$$

We proceed now to integrate (b). By elimination we easily find the following relations:

$$a_{13} = \frac{x_3}{x_2}\,a_{12}, \qquad a_{23} = \frac{x_2}{x_1}\,a_{12}. \tag{18}$$

The equations involving a_{12}, a_{13}, a_{23} give, on integrating, keeping account of (18),

$$a_{12} = x_1\,x_2\,\varphi_2, \qquad a_{13} = x_1\,x_3\,\varphi_2, \qquad a_{23} = x_2\,x_3\,\varphi_2, \tag{19}$$

φ_2 being an arbitrary function of $\sqrt{x_1^2+x_2^2+x_3^2}$ and x_0. We also find the relations

$$\begin{aligned} a_{11}-a_{22} &= (x_1^2-x_2^2)\,\varphi_2, \qquad a_{22}-a_{33} = (x_2^2-x_3^2)\,\varphi_2,\\ a_{33}-a_{11} &= (x_3^2-x_1^2)\,\varphi_2. \end{aligned} \tag{20}$$

Integrating the equations in a_{11}, a_{22} and a_{33} we have

$$a_{11} = \varphi_3 + x_1^2\,\varphi_2, \qquad a_{22} = \varphi_3 + x_2^2\,\varphi_2, \qquad a_{33} = \varphi_3 + x_3^2\,\varphi_2.$$

We have thus obtained the following quadratic form,

$$ds^2 = \varphi_1\,dx_0^2 - R^2\,\varphi_2\left[\frac{x_1\,dx_1+x_2\,dx_2+x_3\,dx_3}{R}\right]^2 - \varphi_3\,[dx_1^2+dx_2^2+dx_3^2],$$

where $R = \sqrt{x_1^2+x_2^2+x_3^2}$. Introducing spherical coördinates

$$x_1 = R\sin\theta\cos\varphi, \qquad x_2 = R\sin\theta\sin\varphi, \qquad x_3 = R\cos\theta,$$

we have, remembering that φ_2 and φ_3 are arbitrary functions of R and x_0,

$$ds^2 = \varphi_1\,dx_0^2 - (\varphi_3+\varphi_2)\,dR^2 - R^2\,\varphi_3\,(d\theta^2+\sin^2\theta\,d\varphi^2). \tag{21}$$

The group G_3 becomes, on introducing the new variables,

$$\begin{aligned} U_1 &= \sin\varphi\,\frac{\partial f}{\partial\theta} + \cot\theta\cos\varphi\,\frac{\partial f}{\partial\varphi},\\ U_2 &= \cos\varphi\,\frac{\partial f}{\partial\theta} - \cot\theta\sin\varphi\,\frac{\partial f}{\partial\varphi}, \qquad U_3 = \frac{\partial f}{\partial\varphi}. \end{aligned} \tag{22}$$

This group is transitive considered as a group in the variables θ and φ, and the variable R does not appear, as was to be expected according to a theorem by Fubini.* We have thus proved the

THEOREM IV. *A necessary and sufficient condition that the space* (13) *shall be reducible to the form* (21) *is that it shall admit the group G_3 as a complete group of motions.*

For the purpose of further specialization we shall consider a few invariants that play an important rôle in the classification of 3- and 4-spaces and also in the general relativity theory.

5. The total curvature of S_4 (curvature scalar) is given by the formula

$$R = \sum_{h,i}^{0\cdots 3} g^{hi} R_{hi}, \qquad R_{hi} = \sum_{p}^{0\cdots 3} \{hp, ip\}. \tag{23}$$

If instead of the symbols $\{hp, ip\}$ we introduce the Riemannian symbols (hp, iq), we have

$$\{hp, ip\} = \sum_{p,q}^{0\cdots 3} g^{pq}(hp, iq) \qquad R = \sum_{p,q}^{0\cdots 3} g^{pq}(hp, iq),$$

where†

$$(hp, iq) = \frac{1}{2}\left[\frac{\partial^2 g_{hq}}{\partial x_i \partial x_p} + \frac{\partial^2 g_{ip}}{\partial x_h \partial x_q} - \frac{\partial^2 g_{hi}}{\partial x_p \partial x_q} - \frac{\partial^2 g_{pq}}{\partial x_h \partial x_i}\right] + \sum_{l,m}^{0\cdots 3} g^{lm}\left(\begin{bmatrix} h\ q \\ m \end{bmatrix}\begin{bmatrix} i\ p \\ l \end{bmatrix} - \begin{bmatrix} h\ i \\ m \end{bmatrix}\begin{bmatrix} p\ q \\ l \end{bmatrix}\right),$$

and the quantities g^{pq} are the co-factors of g_{pq} divided by g. We now define the following expressions:

$$G_{ih} = \frac{1}{2} g_{ih} R - R_{ih}, \tag{24}$$

and, introducing the mixed forms G_i^h, we put

$$G_i^h = \sum_j g^{hj} G_{ji}. \tag{25}$$

We shall also recall here that for the empty space in an Einstein solar field we must have $R_{ik} = 0$, or, what is the same thing, $G_i^h = 0$. Calculating the curvature tensors R_{ik} for the space (21) which we write in the form

$$-ds^2 = \varphi_2 dR^2 + \varphi_3 (d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_1 dt^2 \tag{21'}$$

---

* L. Bianchi, loc. cit., pp. 517–518. See also Fubini's memoir in vol. 3 of Annali di Matematica.

† The non-vanishing Riemannian symbols (hp, iq) are given on p. 238, equations (58).

we have

$$g_{00} = -\varphi_1, \quad g_{11} = \varphi_2, \quad g_{22} = \varphi_3, \qquad g_{33} = \varphi_3 \sin^2\theta,$$

$$g^{11} = \frac{1}{\varphi_2}, \quad g^{22} = \frac{1}{\varphi_3}, \quad g^{33} = \frac{1}{\varphi_3 \sin^2\theta}, \quad g^{00} = -\frac{1}{\varphi_1};$$

$$\begin{aligned} (26a) \qquad & R_{12} = g^{33}(13,23) + g^{00}(10,20), \qquad R_{13} = g^{22}(12,32) + g^{00}(10,30),\\ & R_{23} = g^{11}(21,31) + g^{00}(20,30), \qquad R_{10} = g^{22}(12,02) + g^{33}(13,03),\\ & R_{20} = g^{11}(21,01) + g^{33}(23,03), \qquad R_{30} = g^{11}(31,01) + g^{22}(32,02); \end{aligned}$$

$$(26b) \qquad \begin{aligned} R_{11} &= g^{22}(12,12) + g^{33}(13,13) + g^{00}(10,10),\\ R_{22} &= g^{11}(21,21) + g^{33}(23,23) + g^{00}(10,10),\\ R_{33} &= g^{11}(31,31) + g^{22}(32,32) + g^{00}(30,30),\\ R_{00} &= g^{11}(01,01) + g^{22}(02,02) + g^{33}(30,30). \end{aligned}$$

Calculating the Riemannian symbols (hp, iq) and substituting in these equations we find

$$R_{12} = 0, \quad R_{13} = 0, \quad R_{23} = 0, \quad R_{20} = 0, \quad R_{30} = 0,$$

$$R_{10} = -\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial r\,\partial t} + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_2}{\partial t} + \frac{1}{2\varphi_3^2}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_3}{\partial t} + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial t},$$

$$\begin{aligned} R_{22} = \frac{R_{33}}{\sin^2\theta} = &-\frac{1}{2\varphi_2}\frac{\partial^2\varphi_3}{\partial r^2} + \frac{1}{4\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_3}{\partial r} - \frac{1}{4\varphi_1^2}\frac{\partial\varphi_3}{\partial t}\frac{\partial\varphi_1}{\partial t} + \frac{1}{2\varphi_1}\frac{\partial^2\varphi_3}{\partial t^2}\\ &+ \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial t} - \frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial r}\right] + 1, \end{aligned}$$

$$(27) \qquad \begin{aligned} R_{11} = &-\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial r^2} + \frac{1}{2\varphi_1}\frac{\partial^2\varphi_2}{\partial t^2} - \frac{1}{2\varphi_1}\frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_3}{\partial r}\\ &+ \frac{1}{2\varphi_3^2}\left[\frac{\partial\varphi_3}{\partial r}\right]^2 + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial t} + \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_2}{\partial t}\right]^2 - \frac{1}{4\varphi_1^2}\left[\frac{\partial\varphi_1}{\partial r}\right]^2\\ &+ \frac{1}{4\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial t} - \frac{1}{4\varphi_1\varphi_2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial t}, \end{aligned}$$

$$\begin{aligned} R_{00} = &-\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial t^2} - \frac{1}{2\varphi_2}\frac{\partial^2\varphi_2}{\partial t^2} + \frac{1}{2\varphi_2}\frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_3}{\partial t}\frac{\partial\varphi_1}{\partial t}\\ &+ \frac{1}{2\varphi_3^2}\left[\frac{\partial\varphi_3}{\partial t}\right]^2 + \frac{1}{4\varphi_2^2}\left[\frac{\partial\varphi_2}{\partial t}\right]^2 + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_1}{\partial r} + \frac{1}{4\varphi_1\varphi_2}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_1}{\partial t}\\ &- \frac{1}{\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial r} - \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_1}{\partial r}\right]^2; \end{aligned}$$

$$R = g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} + g^{00} R_{00}, \qquad G_{ii} = \frac{1}{2} g_{ii} R - R_{ii},$$

$$G_i^i = g^{ii} G_{ii} = \frac{1}{2} R - g^{ii} R_{ii}, \qquad G_1^0 = g^{00} G_{01} = -g^{00} R_{10},$$

$$G_i^h = 0, \qquad i, h \mid 1, 0;$$

$$(28) \quad \begin{aligned} G_1^1 &= g^{22} g^{33}(23, 23) + g^{00} g^{22}(20, 20) + g^{33} g^{00}(03, 03), \\ G_2^2 &= g^{11} g^{33}(13, 13) + g^{11} g^{00}(10, 10) + g^{33} g^{00}(03, 03), \\ G_3^3 &= g^{11} g^{22}(12, 12) + g^{11} g^{00}(10, 10) + g^{22} g^{00}(02, 02), \\ G_0^0 &= g^{11} g^{22}(12, 12) + g^{11} g^{33}(13, 13) + g^{22} g^{33}(23, 23); \end{aligned}$$

$$(29) \quad \begin{aligned} G_1^1 = {} & \frac{1}{\varphi_3} - \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2 - \frac{1}{2\varphi_3\varphi_1^2}\frac{\partial \varphi_3}{\partial t}\frac{\partial \varphi_1}{\partial t} \\ & - \frac{1}{2\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_1}{\partial r} + \frac{1}{\varphi_1\varphi_3}\frac{\partial^2 \varphi_3}{\partial t^2}, \\ G_0^0 = {} & \frac{1}{\varphi_3} + \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 + \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2 + \frac{1}{2\varphi_2^2\varphi_3}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r} \\ & + \frac{1}{2\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t} - \frac{1}{\varphi_2\varphi_3}\frac{\partial^2 \varphi_3}{\partial r^2}, \\ G_2^2 = G_3^3 = {} & \frac{1}{2\varphi_1\varphi_2}\frac{\partial^2 \varphi_2}{\partial t^2} - \frac{1}{2\varphi_1\varphi_2}\frac{\partial^2 \varphi_1}{\partial r^2} - \frac{1}{2\varphi_3\varphi_2}\frac{\partial^2 \varphi_3}{\partial r^2} \\ & + \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_2^2}\left[\frac{\partial \varphi_2}{\partial t}\right]^2 + \frac{1}{4\varphi_2\varphi_1^2}\left[\frac{\partial \varphi_1}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2 \\ & + \frac{1}{4\varphi_3\varphi_2^2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r} - \frac{1}{4\varphi_1^2\varphi_2}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_1}{\partial t} - \frac{1}{4\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_1}{\partial r} \\ & - \frac{1}{4\varphi_1^2\varphi_3}\frac{\partial \varphi_3}{\partial t}\frac{\partial \varphi_1}{\partial t} + \frac{1}{4\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t} + \frac{1}{4\varphi_1\varphi_2^2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_1}{\partial r} \\ & + \frac{1}{2\varphi_1\varphi_3}\frac{\partial^2 \varphi_3}{\partial t^2}, \\ G_1^0 = {} & \frac{1}{\varphi_1}\left[\frac{1}{\varphi_3}\frac{\partial^2 \varphi_3}{\partial r\,\partial t} - \frac{1}{2\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial r} - \frac{1}{2\varphi_3^2}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_3}{\partial t} - \frac{1}{2\varphi_1\varphi_3}\frac{\partial \varphi_1}{\partial r}\frac{\partial \varphi_3}{\partial t}\right]. \end{aligned}$$

It is significant that these mixed tensors do not contain any of the variables φ, θ, while R_{33} contains θ.

6. We shall now suppose that the space (21′) admits a one-parameter group whose infinitesimal symbol is of the form

$$G_1: \qquad Uf = \xi_0 \frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial r},$$

where ξ_0 and ξ_1 are functions of r and t. In order that this shall be the case, the functions φ_1, φ_2 and φ_3 must satisfy certain conditions which we shall now proceed to find. The Killing equations (4) are in this case

$$
\begin{aligned}
&\xi_0 \frac{\partial \varphi_2}{\partial t} + \xi_1 \frac{\partial \varphi_2}{\partial r} + 2\varphi_2 \frac{\partial \xi_1}{\partial r} = 0, \qquad \xi_0 \frac{\partial \varphi_1}{\partial t} + \xi_1 \frac{\partial \varphi_1}{\partial r} + 2\varphi_1 \frac{\partial \xi_0}{\partial t} = 0, \\
&\xi_0 \frac{\partial \varphi_3}{\partial t} + \xi_1 \frac{\partial \varphi_3}{\partial r} = 0, \qquad \varphi_2 \frac{\partial \xi_1}{\partial t} - \varphi_1 \frac{\partial \xi_0}{\partial r} = 0.
\end{aligned}
\tag{30}
$$

The third equation shows that φ_3 must be an invariant of the group or a constant. If φ_3 is not a constant, we put

$$
\xi_0 = -\lambda \frac{\partial \varphi_3}{\partial r}, \qquad \xi_1 = \lambda \frac{\partial \varphi_3}{\partial t}.
\tag{31}
$$

Substituting the values of ξ_0, ξ_1, and their derivatives obtained from (31) in (30) we have

$$
\begin{aligned}
-2\frac{\partial \varrho}{\partial r} &= \frac{\dfrac{\partial \varphi_2}{\partial r}\dfrac{\partial \varphi_3}{\partial t} - \dfrac{\partial \varphi_2}{\partial t}\dfrac{\partial \varphi_3}{\partial r} + 2\varphi_2 \dfrac{\partial^2 \varphi_3}{\partial r\,\partial t}}{\varphi_2 \dfrac{\partial \varphi_3}{\partial t}}, \\
-2\frac{\partial \varrho}{\partial t} &= \frac{\dfrac{\partial \varphi_1}{\partial t}\dfrac{\partial \varphi_3}{\partial r} - \dfrac{\partial \varphi_1}{\partial r}\dfrac{\partial \varphi_3}{\partial t} + 2\varphi_1 \dfrac{\partial^2 \varphi_3}{\partial r\,\partial t}}{\varphi_1 \dfrac{\partial \varphi_3}{\partial r}},
\end{aligned}
\tag{32}
$$

$$
-\frac{\partial \varrho}{\partial t}\varphi_2 \frac{\partial \varphi_3}{\partial t} - \frac{\partial \varrho}{\partial r}\varphi_1 \frac{\partial \varphi_3}{\partial r} = \varphi_1 \frac{\partial^2 \varphi_3}{\partial r^2} + \varphi_2 \frac{\partial^2 \varphi_3}{\partial t^2},
\tag{33}
$$

where $\varrho = \log \lambda$. Taking account of (29), these equations may be written

$$
\begin{aligned}
-2\frac{\partial \varrho}{\partial r} &= \frac{\partial}{\partial r}\log \varphi_1 \varphi_2 \varphi_3 + \frac{2\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial t}}, \\
-2\frac{\partial \varrho}{\partial t} &= \frac{\partial}{\partial t}\log \varphi_1 \varphi_2 \varphi_3 + \frac{2\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial r}},
\end{aligned}
\tag{32'}
$$

$$
\varphi_1 \varphi_2 \varphi_3 (G_1^1 - G_0^0) - \left[\frac{\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial t}} \cdot \varphi_1 \frac{\partial \varphi_3}{\partial r} + \frac{\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial r}} \cdot \varphi_2 \frac{\partial \varphi_3}{\partial t}\right] = 0.
\tag{33'}
$$

The conditions which must be satisfied by the functions φ_1, φ_2 and φ_3 are therefore, besides (33'), the following:

$$(34)\qquad \frac{\partial}{\partial t}\left[\frac{\varphi_1\,\varphi_3\,G_1^0}{\dfrac{\partial\varphi_3}{\partial t}}\right] = \frac{\partial}{\partial r}\left[\frac{\varphi_1\,\varphi_3\,G_1^0}{\dfrac{\partial\varphi_3}{\partial r}}\right].$$

We now put

$$(34')\qquad \frac{\varphi_1\,\varphi_3\,G_1^0}{\dfrac{\partial\varphi_3}{\partial t}} = \frac{\partial\Psi}{\partial r},\qquad \frac{\varphi_1\,\varphi_3\,G_1^0}{\dfrac{\partial\varphi_3}{\partial r}} = \frac{\partial\Psi}{\partial t},$$

from which we derive the differential equation for Ψ,

$$\frac{\partial\varphi_3}{\partial t}\,\frac{\partial\Psi}{\partial r} - \frac{\partial\varphi_3}{\partial r}\,\frac{\partial\Psi}{\partial t} = 0.$$

Hence, Ψ must be an arbitrary function of φ_3 or else a constant. The conditions (34) and (33') may now be written

$$(35)\qquad \varphi_1\,\varphi_3\,G_1^0 = \Psi'(\varphi_3)\,\frac{\partial\varphi_3}{\partial r}\,\frac{\partial\varphi_3}{\partial t},$$

$$(35')\qquad \varphi_1\,\varphi_2\,\varphi_3\,(G_1^1 - G_0^0) = \Psi'(\varphi_3)\left[\varphi_1\left(\frac{\partial\varphi_3}{\partial r}\right)^2 + \varphi_2\left(\frac{\partial\varphi_3}{\partial t}\right)^2\right].$$

These conditions being satisfied, the corresponding G_1 has the form

$$Uf = -\frac{\dfrac{\partial\varphi_3}{\partial r}}{e^{\Psi}\sqrt{\varphi_1\,\varphi_2\,\varphi_3}}\cdot\frac{\partial f}{\partial t} + \frac{\dfrac{\partial\varphi_3}{\partial t}}{e^{\Psi}\sqrt{\varphi_1\,\varphi_2\,\varphi_3}}\cdot\frac{\partial f}{\partial r}.$$

We have then

The necessary and sufficient conditions that a centro-symmetric space with linear element

$$(21')\qquad -ds^2 = \varphi_2\,dr^2 + \varphi_3\,(d\theta^2 + \sin^2\theta\,d\varphi^2) - \varphi_1\,dt^2,$$

φ_1, φ_2 and φ_3 being arbitrary functions of r and t and φ_3 not a constant, shall admit a one-parameter group of motions G_1 are

$$(35)\qquad \varphi_1\,\varphi_3\,G_1^0 = \Psi'\cdot\frac{\partial\varphi_3}{\partial r}\,\frac{\partial\varphi_3}{\partial t},$$

$$(35')\qquad \varphi_1\,\varphi_2\,\varphi_3\,(G_1^1 - G_0^0) = +\Psi'\left[\varphi_1\left(\frac{\partial\varphi_3}{\partial r}\right)^2 + \varphi_2\left(\frac{\partial\varphi_3}{\partial t}\right)^2\right],$$

where Ψ is an arbitrary function of φ_3, or a constant.

But we know that by a proper choice of variables this group may always be reduced to the form $\partial f/\partial t$ (Theorem II). The transformation $T = T(r, t)$, $R = R(r, t)$, where R and T satisfy the two partial differential equations

$$(36) \qquad \xi_0 \frac{\partial T}{\partial t} + \xi_1 \frac{\partial T}{\partial r} = 1, \qquad \xi_0 \frac{\partial R}{\partial t} + \xi_1 \frac{\partial R}{\partial r} = 0$$

(solved by two quadratures), will carry (21′) into the form

$$(37) \qquad -ds^2 = \varphi_2\, dr^2 + \varphi_3 (d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2,$$

where φ_1, φ_2 and φ_3 are functions of r alone. It should be noted that the transformation may always be so chosen as to preserve the orthogonality of the space $S(r, \theta, \varphi)$ to the time-axis. We have therefore

THEOREM V. *The necessary and sufficient conditions that a centro-symmetric space* (21′) *shall be reducible to the static form are given by the equations* (35) *and* (35′).

If we choose the arbitrary function $\Psi = \text{const.}$, we have, as a corollary of the above theorem,

If $G_1^1 = G_0^0$ *and* $G_1^0 = 0$, *the centro-symmetric space* (21′) *is reducible to the static form.*

Since for an Einstein solar field we have $G_{ik} = 0$ it follows that *the Einstein solar field is necessarily static.**

For a class of spaces of importance in relativity theory the generalized Einstein equations hold: $G_{ik} = \lambda g_{ik}$, where λ is constant. We have therefore $G_1^0 = 0$, $G_1^1 - G_0^0 = 0$. Hence, *these spaces are also necessarily static.* Examples are De Sitter's cosmological space and the gravitational field of an electron.†

The case $\varphi_3 = \text{const.}$ needs special treatement, since

$$\frac{\partial \varphi_3}{\partial r} = \frac{\partial \varphi_3}{\partial t} = 0.$$

It will be convenient to transform (21′) into the form

$$(21'') \qquad -ds^2 = \varphi_1 (dr^2 - dt^2) + \varphi_3 (d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Integrating equations (30) on the hypotheses $\varphi_1 = \varphi_2$ and $\varphi_3 = \text{const.}$ we find

$$\frac{\partial \xi_1}{\partial r} = \frac{\partial \xi_0}{\partial t}, \qquad \frac{\partial \xi_1}{\partial t} = \frac{\partial \xi_0}{\partial r}, \qquad \varphi_1 = \varphi_2 = \varphi(r^2 - t^2);$$

* For a different proof of this proposition see G. D. Birkhoff, loc. cit., pp. 253–256.

† A. S. Eddington, *The Mathematical Theory of Relativity*, Cambridge University Press, 1923, p. 100 and p. 185.

hence

$$\xi_0 = \Phi(r+t) + \Psi(r-t), \qquad \xi_1 = \Phi(r+t) - \Psi(r-t).$$

Without loss of generality we may put $\xi_1 = t$, $\xi_0 = r$, and the group is

$$Uf = r\frac{\partial f}{\partial t} + t\frac{\partial f}{\partial r}.$$

The integration of (36) will yield the transformation

$$r+t = R \cdot e^{T}, \qquad r-t = R \cdot e^{-T},$$

which will carry (21″) into the static form. We also find

$$G_1^0 = 0, \qquad G_1^1 - G_0^0 = \frac{1}{\varphi_3} - \frac{1}{\varphi_3} = 0.$$

To this class belongs the space discussed by Levi-Civita (p. 238, footnote). This space is also necessarily static.

The case where φ_3 is a function of r alone, say $\varphi_3 = r^2$, is not special, since a transformation

$$R^2 = \varphi_3, \qquad T = \Phi(r, t)$$

will carry (21′) into

(21‴) $$-ds^2 = -\bar{\varphi}_1\, dT^2 + \bar{\varphi}_2\, dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

$G_1^0 = 0$, since $\partial\varphi_3/\partial T = 0$ and (34′) shows that Ψ is a function of R alone. The condition (35′) becomes

(35″) $$\bar{\varphi}_2(G_1^1 - G_0^0) = 4\Psi'(R).$$

7. We shall suppose that the linear element (21′) admits the groups G_1 and G_3, all the transformations of which form a group G_4, and that it has been reduced to the form (37). If φ_3 is not constant, the transformation

$$R = \sqrt{\varphi_3}, \qquad T = t, \qquad \varphi_2 = 1 + \bar{\varphi}_2$$

will reduce (37) to the form

(38) $$ds^2 = \varphi_1\, dt^2 - (1+\varphi_2)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

admitting the group of motions G_4.

We have now $R_{ik} = 0$, $i \neq k$, and the mixed tensors G_i^i, which we shall substitute for the tensors R_{ii}, become (equations (29))

$$G_1^1 = \frac{1}{r^2(1+\varphi_2)}\left[\varphi_2 - \frac{r\varphi_1'}{\varphi_1}\right], \quad G_0^0 = \frac{1}{r^2(1+\varphi_2)}\left[\varphi_2 + \frac{r\varphi_2'}{1+\varphi_2}\right],$$

$$(39)\quad G_2^2 = G_3^3 = \frac{-\varphi_1''}{2\varphi_1(1+\varphi_2)} + \frac{\varphi_2'\varphi_1'}{4\varphi_1(1+\varphi_2)^2} + \frac{\varphi_2'}{2(1+\varphi_2)^2 r} + \frac{\varphi_1'^2}{4(1+\varphi_2)\varphi_1^2} - \frac{\varphi_1'}{2\varphi_1(1+\varphi_2)r}.$$

It appears from these equations that while $G_1^0 = 0$, the same is not true of the difference $G_1^1 - G_0^0$. We find

$$(40)\qquad G_1^1 - G_0^0 = -\left[\varphi_1' + \frac{\varphi_1}{1+\varphi_2}\varphi_2'\right]\frac{1}{r(1+\varphi_2)\varphi_1}$$

which vanishes if, and only if, $\varphi_1(1+\varphi_2) = c$. This ist he "Leithypothese" of Kottler* which may be stated in the form of an equation as follows:

$$(41)\quad g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & 0 \\ g_{12} & g_{22} & g_{23} & 0 \\ g_{13} & g_{23} & g_{33} & 0 \\ 0 & 0 & 0 & -g_{00} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & -c^2 \end{vmatrix} = |a_{ik}|\,(-c^2),$$

where the a's are the coefficients of the linear element of a euclidean space and c is the velocity of light in "empty" space. In fact, we have

$$g = -\varphi_1(1+\varphi_2)r^4\sin^2\theta = r^4\sin^2\theta(-c^2),$$

or

$$\varphi_1(1+\varphi_2) = c^2.$$

Consider the three sub-spaces $t = c_1$, $\varphi = c_2$, $\theta = c_3$, having the respective linear elements

$$\begin{aligned} &S_3: && ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),\\ (42)\quad &S_3^1: && ds^2 = (1+\varphi_2)\,dr^2 + r^2\,d\theta^2 - \varphi_1\,dt^2,\\ &S_3^2: && ds^2 = (1+\varphi_2)\,dr^2 + r^2\sin^2\theta\, d\varphi^2 - \varphi_1\,dt^2. \end{aligned}$$

* F. Kottler, *Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie*, Annalen der Physik, vol. 56, pp. 401-462. Kottler considers this hypothesis as a substitute in general relativity for the hypothesis of the constancy of the velocity of light in Minkowski's mechanics. The hypothesis holds for an Einstein solar field, but not necessarily for spaces with a different mass-distribution. It does not hold for Einstein's cosmological space, although it does for that of De Sitter.

We shall calculate the principal Riemannian curvatures for each of them. We find*

$$
(43)\quad
\begin{aligned}
&S_3: && K_1 = \frac{\varphi_2}{r^2(1+\varphi_2)}, \qquad K_2 = K_3 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},\\
&S_3^1: && K_1^1 = \frac{-\varphi_1''}{2\varphi_1(1+\varphi_2)} + \frac{\varphi_2'\varphi_1'}{4\varphi_1(1+\varphi_2)^2} + \frac{\varphi_1'^2}{4\varphi_1^2(1+\varphi_2)},\\
& && K_2^1 = \frac{\varphi_1'}{4\varphi_1^2(1+\varphi_2)}, \qquad K_3^1 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},\\
&S_3^2: && K_1^2 = K_1, \quad K_2^2 = K_2^1, \quad K_3^2 = K_3^1 = K_2 = K_3.
\end{aligned}
$$

Comparing these results with (40) we have, at once,

$$
(44)\quad
\begin{aligned}
G_0^0 &= K_1 + K_2 + K_3,\\
G_2^2 &= G_3^3 = K_1^1 + K_2^1 + K_3^1 = K_1^2 + K_2^2 + K_3^2.
\end{aligned}
$$

Since in S_3 we have $K_2 = K_3$, the principal trihedron with respect to any point P has one direction completely determined, namely the direction of the tangent to the r-line, while the other two directions through P, normal to it and to each other, may be chosen arbitrarily in the tangent plane to the surface $r = \text{const}$. The space S_3 can therefore rotate freely about the geodesic r.†

This is not the case with the spaces S_3^1 and S_3^2, since the three principal curvatures are in general unequal. If, however, $K_2^1 = K_3^1$, S_3^1, and also S_3^2 will have the same property as S_3: At a point P the principal direction along the r-line will be determinate, while the other two are arbitrary in the tangent planes to the surface $r = \text{const}$. The condition $K_2^1 = K_3^1$ imposes therefore a certain symmetry on the time space S_4, namely

The three subspaces S_3, S_3^1, S_3^2 possess, at any generic point P, "rotational mobility" about the geodesic r-line which passes through the point.

It should be noted that, in the case of the subspaces S_3^1 and S_3^2, a "rotation" about the r-line means a "hyperbolic" rotation, since these spaces have an indefinite quadratic form as line-element. If with Minkowsky we put $it = t$, the hyperbolic rotation becomes ordinary euclidean.

---

* L. Bianchi, *Lezione di Geometria Differenziale*, vol. 1, pp. 365–358, and also *Lezioni sulla Teoria dei Gruppi*, pp. 546–547 by the same author.

† It should be observed in this connection that when we say "rotate freely about the geodesic r" this does not imply that the r-line is an axis. It would be a geodesic axis only if $K_2 = K_3 = 0$, that is, if φ_2 is a constant. The transformation $\bar{r} = \int \sqrt{1+\varphi_2}\, dr$ will make r a geodesic axis.

Since $K_2^1 = K_3^1$ is equivalent to the condition $G_1^1 = G_0^0$, or to the condition

$$\varphi_1(1+\varphi_2) = \text{const.}, \tag{45}$$

we see that Kottler's *"Leithypothese" is equivalent to the assumption of free mobility of the subspaces S_3, S_3^1, S_3^2 about the r-line.*

With this assumption the linear element takes the form

$$-ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \frac{c}{1+\varphi_2}\,dt^2 \tag{46}$$

and we also have, from (40),

$$G_1^1 = G_0^0 = K_1 + K_2 + K_3,\quad G_2^2 = G_3^3 = K_1^1 + K_2^1 + K_3^1 = K_1^2 + K_2^2 + K_3^2. \tag{47}$$

8. We shall specialize further by assuming various values for the sum of the three principal curvatures of S_3, which is the space part of the time-space S_4.

(*a*) $K_1 + K_2 + K_3 = 0$. We have from (40)

$$\varphi_2\varphi_1 - r\varphi_1' = 0,$$

or, since $\varphi_2 = c^2/\varphi_1 - 1$,

$$r\varphi_1 = c^2 r + C.$$

Putting $C = -\alpha c^2$ we have

$$\varphi_1 = c^2\left(1 - \frac{\alpha}{r}\right), \qquad 1 + \varphi_2 = \frac{1}{1 - \dfrac{\alpha}{r}},$$

which is Schwarzschild's solution for the line-element of a static and stationary space with centro-symmetric mass-distribution, viz.:

$$-ds^2 = \frac{dr^2}{1 - \dfrac{\alpha}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{\alpha}{r}\right)dt^2. \tag{48}$$

This space is characterized by the following three geometrical properties:

1. It admits the group of motions G_4 (Assumption L).
2. The three subspaces S_1, S_2, S_3 possess rotational mobility about the r-line (Assumption M).
3. The sum of the three principal Riemannian curvatures of the space S_3 is zero (Assumption N).

Assumptions L, M, and N* are equivalent to the physical assumption that in the gravitational field outside the mass the tensors T_{ik} all vanish. In fact Einstein's equations are

$$(49) \qquad G_{ik} = \varkappa T_{ik} = 0, \qquad \varkappa = \frac{8\pi k}{c^4},$$

where k is the Newtonian gravitation constant. But from (40) we have

$$G_i^k = g^{ik} G_{ik} = 0, \qquad i \neq k,$$

and since $\sum_1^3 K_i = 0$ and $K_2^1 = K_3^1$, it follows that $G_1^1 = G_0^0 = 0$. If the values of φ_1 and $1+\varphi_2$ are substituted in the expression for G_2^2 and G_3^3 we find that they vanish also.

It will be noted that the principal Riemannian curvatures of the three subspaces S_3, S_3^1, S_3^2 are

$$K_1 = K_1^1 = K_1^2 = \frac{\alpha}{r^3}, \qquad K_2 = K_2^1 = K_2^2 = K_3 = K_3^1 = K_3^2 = -\frac{\alpha}{2r^3}.$$

If, as has been suggested by Cesàro,† we put

$$K = \frac{1}{3}\sum K_i, \qquad K^1 = \frac{1}{3}\sum K_i^1, \qquad K^2 = \frac{1}{3}\sum K_i^2,$$

and define K, K^1, K^2 as the mean curvatures of the respective spaces, we have

THEOREM VI. *The subspaces S_3, S_3^1 and S_3^2 of a static time-space* (48) *with centro-symmetric mass distribution have their mean curvatures equal to zero, and the three principal curvatures of any subspace are respectively equal to the corresponding principal curvatures of any other subspace.*

(b) $K_1+K_2+K_3 = \text{const.} = 3/R^2$. We have from (43)

$$\frac{\varphi_2\varphi_1 - r\varphi_1'}{r^2c^2} = \frac{3}{R^2},$$

or

$$c^2 - \varphi_1 - r\varphi_1' = \frac{3c^2r^2}{R^2};$$

* L, M and N are also equivalent to postulates I—V of Eisenhart's paper *The permanent gravitational field in the Einstein theory*, Annals of Mathematics, ser. 2, vol. 22, No. 2; December, 1920.

† Ernesto Cesàro, *Lezioni di Geometria Intrinsica*, Napoli, 1896, p. 223.

integrating, we have

$$(50) \qquad \varphi_1 = \frac{C}{r} + c^2\left(1 - \frac{r^2}{R^2}\right).$$

Let $C = 0$; the linear element of S_4 is

$$(51) \qquad -ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{r^2}{R^2}\right) dt^2.$$

This solution applies to the space inside of a homogeneous sphere of matter provided the inertial density (Trägheitsdichte) is assumed to be zero.* If we calculate the Riemannian symbols (ik, rh), we find

$$(ik, ik) = \frac{1}{R^2} a_{ii} a_{kk}, \qquad (ik, rh) = 0, \quad i \neq r, \quad k \neq h,$$

which means that the curvature $K_0 = 1/R^2$ (L. Bianchi, *Lezioni*, vol. I, p. 344). The group of S_4 is therefore the maximum G_{10} of non-euclidean rotations and translations, and that of S_3 is the corresponding G_6. G_6 does not belong to S_4, while the group G_4 remains as a subgroup of G_{10}. The space (51) is usually referred to as *De Sitter's space.*

(*c*) If in (50) we let the constant C differ from zero, we may put $C = c^2 b^3/R^2$ and we have

$$(52) \quad \varphi_1 = c^2\left[1 - \frac{r^2}{R^2} + \frac{b^3}{R^2}\cdot\frac{1}{r}\right], \qquad 1 + \varphi_2 = \frac{1}{1 - \dfrac{r^2}{R^2} + \dfrac{b^3}{R^2}\cdot\dfrac{1}{r}}.$$

The space to which this solution applies is that of a shell of thickness $a - b$. R is supposed to vary between the limits $b > R < a$.† Since the shell acts on the region outside of it like a Newtonian masspoint m determined by the relation

$$\frac{2km}{c^2} = \frac{a^3 - b^3}{R^2},$$

* H. Weyl, *Raum, Zeit und Materie*, 4th edition, p. 232; F. Kottler, loc. cit., pp. 438–439. Kottler assumes that the cohesion pressure equals the entire energy of the mass, i. e. $\mu = \varepsilon - p = 0$ where ε is the energy and p the cohesion pressure. See also H. Weyl, loc. cit., p. 254, and p. 256.

† F. Kottler, loc. cit., p. 493. He obtains this solution by assuming that the space has no inertial mass, i. e. the cohesion pressure $p = \varepsilon$. Kottler's solution has not been accepted by Einstein, who prefers the one obtained by Schwarzschild (equation 55).

we have, for $R > a$,

$$\varphi_1 = c^2\left[1 - \frac{a^3 - b^3}{R^2}\,\frac{1}{r}\right], \quad 1 + \varphi_2 = \frac{1}{1 - \dfrac{a^3 - b^3}{R^2}\,\dfrac{1}{r}}.$$

(d) We shall consider the case where $K_1 + K_2 + K_3$ is inversely proportional to the fourth power of the "distance" r from the mass-center. We have

$$K = \frac{\varphi_2 \varphi_1 - r\varphi_1}{r^2 c^2} = \frac{\alpha}{r^4}. \tag{53}$$

or, since we are still working under assumption M,

$$c^2 - \varphi_1 - r\varphi_1' = \frac{c^2 \alpha}{r^2}.$$

Integrating we have

$$\varphi_1 = c^2\left(1 - \frac{\beta}{r} + \frac{\alpha}{r^2}\right) \quad \text{and} \quad 1 + \varphi_2 = \frac{1}{1 - \dfrac{\beta}{r} + \dfrac{\alpha}{r^2}}.$$

S_4 has therefore the linear element

$$-ds_2 = \frac{dr^2}{1 - \dfrac{\beta}{r} + \dfrac{\alpha}{r^2}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{\beta}{r} + \frac{\alpha}{r^2}\right) dt^2, \tag{54}$$

which has been obtained by Weyl.* He considers a sphere having a Newtonian mass m and a static charge e. If we put $\beta = 2a$ and $\alpha = ke^2/c^4$, where $a = km/c^2$, we get the identical form due to Weyl. The group of the space (54) is G_4, assumptions L and M hold, while for N is substituted (53). This space, and the two preceding ones, belong to a class of spaces characterized by the property of having the sum of the three principal curvatures of S_3 equal to a function of the distance from the mass-center, assumptions L and M being valid. We find for the value of φ_1

$$\varphi_1 = c^2\left[1 - \frac{1}{r}\int^r F(r)\,dr\right].$$

(e) Let us retain assumption L. Instead of M, i. e. $K_2^1 = K_3^1$, we put

M': $$K_1^1 = K_2^1,$$

* H. Weyl, loc. cit., pp. 236–237.

and let, as before in case (b),

N': $$K_1 + K_2 + K_3 = \frac{3}{R^2}.$$

φ_1 and φ_2 must now satisfy the following differential equations obtained from (42):

$$\frac{\varphi_2}{r^2(1+\varphi_2)} + \frac{\varphi_2'}{r(1+\varphi_2)^2} = \frac{3}{R^2},$$

$$\frac{\varphi_1'}{r\varphi_1} + \frac{\varphi_1'^2}{2\varphi_1^2} + \frac{\varphi_2'\varphi_1'}{2\varphi_1(1+\varphi_2)} - \frac{\varphi_1''}{\varphi_1} = 0.$$

Integrating the first equation, we find

$$1+\varphi_2 = \frac{1}{1-\dfrac{C}{r}-\dfrac{r^2}{R^2}};$$

we let C be equal to zero and substitute the value of φ_2 in the second equation and integrate; the result is

$$\sqrt{\varphi_1} = c\left[\alpha - \frac{\sqrt{1-\dfrac{r^2}{R^2}}}{2}\right],$$

where c and α are integration constants. If we determine the initial value of φ_1 in such a way that for $r = R$ we have

$$\sqrt{\varphi_1} = \frac{3}{2}c\sqrt{1-\frac{a^2}{R^2}},$$

we get

$$\sqrt{\varphi_1} = c\left[\frac{3\sqrt{1-\dfrac{a^2}{R^2}}-\sqrt{1-\dfrac{r^2}{R^2}}}{2}\right],$$

and the linear element of the space S_4 is

$$(55)\qquad -ds^2 = \frac{dr^2}{1-\dfrac{r^2}{R^2}} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$
$$-c^2\left[\frac{3\sqrt{1-\dfrac{a^2}{R^2}}-\sqrt{1-\dfrac{r^2}{R^2}}}{2}\right]^2 dt^2,$$

which is Schwarzschild's solution for the gravitational space within a liquid sphere of radius a. The group of S_3 is obviously G_6, *but G_6 does not belong to S_4*. The condition $C = 0$ is equivalent to

$$P: \qquad K_1 = K_2 = K_3 = \frac{1}{R^2}.$$

S_3 is therefore a space of constant curvature. Assumptions L, M', and P determine completely the Schwarzschild solution (55).

9. **The case** $g_{00} =$ const. If g_{00} is constant, the equations (43) become

$$K_1^1 = K_2^1 = 0, \quad K_1^2 = K_2^2 = 0, \quad K_2 = K_3 = K_3^1 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},$$

$$K_1 = \frac{\varphi_2}{r^2(1+\varphi_2)}.$$

The group of S_4 is the systatic G_4, and, since g_{00} is constant, the transformation $\partial f / \partial t$ is a translation (Theorem II). The space S_4 can therefore rotate freely about the t-line as geodesic axis. We shall consider the following cases:

(a) $K_1 + K_2 + K_3 = 0$. We have

$$\varphi_2(1+\varphi_2) + r\varphi_2' = 0.$$

Integrating we find

$$1 + \varphi_2 = \frac{1}{1 - \dfrac{\alpha}{r}},$$

where α is the integration constant. If $\alpha = 0$, the space is euclidean. $\alpha \neq 0$ does not correspond to any physical space, since the velocity of light is constant.

(b) $K_1 + K_2 + K_4 = 3/R^2$. This case gives us the Einstein-Schrödinger solution for a closed space with incoherent mass in static equilibrium,*

$$(56) \qquad -ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - g_{00}\, dt^2.$$

---

* E. Schrödinger, *Über die Lösungssysteme der allgemeinen kovarianten Gravitationsgleichungen*, Physikalische Zeitschrift, vol. 19 (1918), p. 20. See also F. Kottler, loc. cit., p. 483, and H. Weyl, loc. cit., pp. 252-253.

S_3 is a space of constant curvature $1/R^2$. The group of S_4 is a G_7, namely $\partial f/\partial t$, and G_6 which belongs also to S_3. G_7 is systatic, the systatic spreads being the geodesic t-lines.

A general time-space (21′) which admits the group G_7 can be reduced to the form (56); for since S_4 admits the group G_6 which is a transitive group in S_3, φ_1 must be constant (Theorem III). Hence, the group G_7 completely characterizes the Einstein-Schrödinger solution (56).

10. A somewhat interesting type of centro-symmetric spaces is obtained from the form (38) by assuming $\varphi_3 = \text{const.}$ Let this constant be R^2. The transformation

$$R\theta = \bar{\theta}, \qquad R\varphi = \bar{\varphi}, \qquad \sqrt{1+\varphi_2}\,dr = d\bar{r}$$

will carry the linear element into the form

$$-ds^2 = d\bar{r}^2 + d\bar{\theta}^2 + \sin^2\frac{\bar{\theta}}{R}\,d\bar{\varphi}^2 - \varphi_1\,dt^2. \tag{57}$$

The principal curvatures of the sub-spaces S_3, S_3^1, S_3^2 are

$$K_1 = \frac{1}{R^2} = G_1^1 = G_0^0;\quad K_2 = K_3 = 0;\quad K_1^1 = K_1^2 = 0;\quad K_3^1 = K_3^2 = 0;$$

$$K_2^1 = K_2^2 = -\frac{1}{2\varphi_1}\frac{\partial^2\varphi_1}{\partial \bar{r}^2} + \frac{1}{4\varphi_1^2}\left(\frac{\partial\varphi_1}{\partial\bar{r}}\right)^2 = G_2^2 = G_3^3.$$

S_3 belongs to a type of L. Bianchi's *normal spaces*, namely that one for which the three principal curvatures are constant. If the mean curvature $K = \sum K_i$ is positive, it is said to be of type B.* This is here the case. The group of the space S_4 is G_4 as before, but S_3 admits a 4-parameter group of motions, namely G_3 and the translation $\partial f/\partial \bar{r}$, the latter not belonging to S_4 except when φ_1 is constant.

A notable case is where the mean curvature of the space S_4 is zero. We have then

$$M = \sum_0^3 G_i^i = \frac{2}{R^2} - \frac{1}{\varphi_1}\frac{\partial^2\varphi_1}{\partial\bar{r}^2} + \frac{1}{2\varphi_1^2}\left(\frac{\partial\varphi_1}{\partial\bar{r}}\right)^2 = 0,$$

which integrated gives

$$\sqrt{\varphi_1} = c_1 e^{\bar{r}/R} + c_2 e^{-\bar{r}/R}.$$

* L. Bianchi, *Sugli spazi normali a tre dimenzioni colle curvature principali costanti*, Lincei Rendiconti, ser. 5, vol. 25, 1st semester 1916, pp. 59–63.

This space has been obtained by T. Levi-Civita. He assumes that the physical space S_3 is supplied with an electrical potential $\varphi = -Cr$. The intensity of the force is $|C|\sqrt{g^{11}}$ and is directed normal to the surfaces $r = \text{const.}$*

11. **Class of a centro-symmetric space.** It follows from a theorem due to E. Kasner that the special Einstein 4-dimensional spread (48) cannot be immersed in a flat 5-space. In other words, the linear element (48) is at least of the second class.† The necessary and sufficient conditions that a differential form ds^2 in n-space shall be of the first class are

1. It must be possible to find a doubly symmetric system of quantities b_{ik} (coefficients of the second differential form) such that

$$(a) \qquad (rk, ih) = b_{ri}\, b_{kh} - b_{ki}\, b_{rh}.$$

2. The system b_{ik} must satisfy the differential equations

$$(b) \qquad \frac{\partial b_{ri}}{\partial x_h} - \frac{\partial b_{rh}}{\partial x_i} + \sum_t^{1\cdots n} \left\{ {r\ i \atop t} \right\} b_{ht} - \sum_t^{1\cdots n} \left\{ {r\ h \atop t} \right\} b_{it} = 0.$$

Equations (a) and (b) are known as the generalized equations of Gauss and Codazzi‡ for a euclidean n-space.

Starting with the general centro-symmetric space (21′) we shall proceed to find the conditions (a). Calculating the Riemannian symbols (rk, ih) we find that they all vanish except the following:

$$\begin{aligned}
(12, 12) &= -\frac{1}{2}\frac{\partial^2 \varphi_3}{\partial r^2} + \frac{1}{4\varphi_2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r} + \frac{1}{4\varphi_3}\left(\frac{\partial \varphi_3}{\partial r}\right)^2 + \frac{1}{4\varphi_1}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t};\\
(13, 13) &= (12, 12)\sin^2\theta;\\
(58) \quad (10, 10) &= -\frac{1}{2}\frac{\partial^2 \varphi_2}{\partial t^2} + \frac{1}{2}\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{4\varphi_2}\left(\frac{\partial \varphi_2}{\partial t}\right)^2 - \frac{1}{4\varphi_2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_1}{\partial r}\\
&\quad - \frac{1}{4\varphi_1}\left(\frac{\partial \varphi_1}{\partial r}\right)^2 + \frac{1}{4\varphi_1}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_1}{\partial t};\\
(23, 23) &= \left[\varphi_3 + \frac{1}{4\varphi_1}\left(\frac{\partial \varphi_3}{\partial t}\right)^2 - \frac{1}{4\varphi_2}\left(\frac{\partial \varphi_3}{\partial r}\right)^2\right]\sin^2\theta;
\end{aligned}$$

* T. Levi-Civita, *Realtà fisica di alcuni spazi normali del Bianchi*, Lincei Rendiconti, ser. 5, vol. 25, 1st semester 1917.

† E. Kasner, *The impossibility of Einstein fields immersed in a flat space of five dimensions*, American Journal of Mathematics, vol. 43, pp. 126–129. For the definition of class see Ricci and Levi-Civita, *Méthodes de calcul différentiel absolu*, Mathematische Annalen, vol. 54, p. 160.

‡ L. Bianchi, *Lezioni di Geometria Differenziali*, 2d edition, vol. 1, p. 462.

$$
(58)\quad
\begin{aligned}
(02,02) &= -\frac{1}{2}\frac{\partial^2\varphi_3}{\partial t^2}+\frac{1}{4\varphi_2}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial r}+\frac{1}{4\varphi_3}\left(\frac{\partial\varphi_3}{\partial t}\right)^2\\
&\quad+\frac{1}{4\varphi_1}\frac{\partial\varphi_1}{\partial t}\frac{\partial\varphi_3}{\partial t};\\
(03,03) &= (02,02)\sin^2\theta;\\
(12,02) &= -\frac{1}{2}\frac{\partial^2\varphi_3}{\partial r\,\partial t}+\frac{1}{4\varphi_2}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial r}+\frac{1}{4\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_3}{\partial t}\\
&\quad+\frac{1}{4\varphi_1}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial t};\\
(13,03) &= (12,02)\sin^2\theta.
\end{aligned}
$$

Equations (a) are now

$$
(59)\quad
\begin{gathered}
b_{02}=b_{03}=b_{13}=b_{23}=b_{12}=0,\quad b_{01}b_{22}=(12,02),\\
b_{01}b_{33}=(13,03),\ b_{00}b_{33}=(03,03),\ b_{00}b_{22}=(02,02),\ b_{11}b_{33}=(13,13),\\
b_{22}b_{33}=(23,23),\quad b_{11}b_{22}=(12,12),\quad b_{00}b_{11}-b_{01}^2=(01,01),
\end{gathered}
$$

from which we derive the relations

$$
\frac{b_{22}}{b_{33}}=\frac{(12,12)}{(13,13)}=\frac{(02,02)}{(03,03)}=\frac{(12,02)}{(13,03)},
$$

that is, we have the following two conditions:

$$
(a_1)\qquad (12,12)(03,03)=(02,02)(13,13),
$$

$$
(a_2)\qquad (12,12)(13,03)=(13,13)(12,02),
$$

which are satisfied by (58). We also find the following values for the non-vanishing b's:

$$
b_{00}^2=\frac{(02,02)^2}{D},\ b_{11}^2=\frac{(12,12)^2}{D},\ b_{22}^2=D,\ b_{33}^2=D\sin^4\theta,\ b_{10}^2=\frac{(12,02)^2}{D},
$$

where $D=(23,23)/\sin^2\theta$. Substituting in the last equation of (59), we obtain the condition

$$
(a_3)\qquad (02,02)(13,13)=(01,01)(23,23)+(12,02)(13,03),
$$

which is *not* satisfied for a general space (21′).

Suppose now that the condition (a_3) is satisfied. A rather long, but not difficult, calculation will show that the b's also satisfy the Codazzi equations (b). We have thus proved the

THEOREM VII. *A necessary and sufficient condition that the general centro-symmetric space* (21′) *shall be of the first class is*

$$(a_3) \qquad (02, 02)(13, 13) = (01, 01)(23, 23) + (12, 02)(13, 03).$$

Let the space (21′) be static, and reduced to the form

$$(38) \qquad -ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2.$$

The condition (a_3) becomes

$$\varphi_1'' = \frac{1}{2}\left[\frac{\varphi_1'\varphi_2'}{\varphi_2} + \frac{(\varphi_1')^2}{\varphi_1}\right],$$

which integrated gives

$$(60) \qquad \varphi_2 = \frac{k^2}{4}\,\frac{(\varphi_1')^2}{\varphi_1}.$$

This condition being satisfied, the spread (38) may be represented in a flat 5-space, the coördinates being

$$x_1 = r\sin\theta\sin\varphi, \qquad x_2 = r\sin\theta\cos\varphi, \qquad x_3 = r\cos\theta,$$
$$x_4 = k\sqrt{\varphi_1}\cos\frac{\bar t}{k}, \qquad x_5 = k\sqrt{\varphi_1}\sin\frac{\bar t}{k},$$

where $\bar t = i\,t$. Hence

A necessary and sufficient condition that the static centro-symmetric space (38) *shall be immersible in a flat* 5-*space is*

$$(60) \qquad \varphi_2 = \frac{k^2}{4}\,\frac{(\varphi_1')^2}{\varphi_1}.$$

12. Particular spaces which occur in the theory of relativity and for which the condition (60) is satisfied are the following:

1. De Sitter's space (51); $\varphi_1 = c^2(1 - r^2/R^2)$, $1 + \varphi_2 = 1/(1 - r^2/R^2)$.
2. All spaces for which $g_{00} = \varphi_1$ is constant (Einstein's cosmic space).
3. The Schwarzschild solution for a space inside a liquid sphere of radius a_0 (equation (55)). The coördinates of the 4-spread are

$$x_1 = r\sin\theta\sin\varphi, \qquad x_2 = r\sin\theta\cos\varphi, \qquad x_3 = r\cos\theta,$$

$$x_4 = r\left[3a_0 - \sqrt{1 - \frac{r^2}{R^2}}\right]\cos\frac{c\bar t}{2R}, \quad x_5 = r\left[3a_0 - \sqrt{1 - \frac{r^2}{R^2}}\right]\sin\frac{c\bar t}{2R}.$$

This surface is a 4-dimensional torus generated by revolving the 3-dimensional sphere

$$(x_4^2 - 3\,a_0\,r)^2 + x_1^2 + x_2^2 + x_3^2 = r^2$$

in such a way that the center describes the circle $x_4^2 + x_5^2 = (3\,a_0 r)^2$. The equation of the torus is

$$\left[\sum_1^3 x_i^2 - r^2(1+9\,a_0^2)\right]^2 = 36\,a_0^2\,r^2\left[r^2 - \sum_1^3 x_i^2\right].$$

Since $a_0 > \frac{1}{3}$,* we have $3\,a_0\,r > r$ so that the generating sphere does not intersect the x_1, x_2, x_3 and x_4-axes. The space inside the liquid sphere in S_3 is represented by a 3-dimensional spherical cap on the generating sphere defined by the limit $\sum x_i^2 < a_0^2 < r^2(1-a_0^2)$, and the corresponding time-space by a 4-dimensional zone on the torus generated by the revolution of the cap.

13. We have seen that the space of an Einstein solar field is at least of the second class. We shall prove the following general

THEOREM VIII. *The linear element* (21′) *for which the condition* (a_3) *is not satisfied is of the second class.*

The transformation

$$\varphi_3(r,t) = R^2, \qquad \psi(r,t) = T$$

will carry (21′) into the form

$$(21''') \qquad -ds^2 = -\bar\varphi_1\,d\,T^2 + \bar\varphi_2\,d\,R^2 + R^2(d\,\theta^2 + \sin^2\theta\,d\,\varphi^2)$$

or

$$(61) \qquad \begin{aligned} -ds^2 = d\sigma^2 + ds_0^2 = -\bar\varphi_1\,d\,T^2 + (\bar\varphi_2 - 1)\,d\,R^2 \\ + (d\,R^2 + R^2\,d\,\theta^2 + R^2\sin^2\theta\,d\varphi^2). \end{aligned}$$

Consider the linear element

$$(62) \qquad d\sigma^2 = -\bar\varphi_1\,d\,T^2 + (\bar\varphi_2 - 1)\,d\,R^2.$$

By the general theory of surfaces, if the Gaussian curvature of the 2-spread (62) differs from zero, it is always possible to find three functions x_4, x_5, and x_6 of R and T such that

$$d\,x_4^2 + d\,x_5^2 + d\,x_6^2 = d\sigma^2,$$

* H. Weyl, loc. cit., p. 242; $a_0 = r_0$ and $r = a$ in Weyl's notation.

and the linear element (61) may be put in the form

$$-ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2,$$

where

$$x_1 = R\sin\theta\sin\varphi, \qquad x_2 = R\sin\theta\cos\varphi, \qquad x_3 = R\cos\theta.$$

The 2-spread whose coördinates are x_4, x_5, and x_6 we shall call *the auxiliary surface.*

If the Gaussian curvature of the 2-spread (62) is zero, it will be possible to express $d\sigma^2$ in the form $d\sigma^2 = dx_4^2 + dx_5^2$, that is, the linear element (61) is of the first class. We find

$$K = \frac{1}{2\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\left\{\frac{\partial}{\partial T}\left(-\frac{1}{\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\cdot\frac{\partial\bar\varphi_2}{\partial T}\right) + \frac{\partial}{\partial R}\left(\frac{1}{\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\cdot\frac{\partial\bar\varphi_1}{\partial R}\right)\right\},$$

or, putting K equal to zero and simplifying,

$$\frac{\partial^2\bar\varphi_1}{\partial R^2} - \frac{\partial^2\bar\varphi_2}{\partial T^2} + \frac{1}{2\bar\varphi_1}\frac{\partial\bar\varphi_2}{\partial T}\frac{\partial\bar\varphi_1}{\partial T} - \frac{1}{2\bar\varphi_1}\left(\frac{\partial\bar\varphi_1}{\partial R}\right)^2 + \frac{1}{2(\bar\varphi_2 - 1)}\cdot\left\{\left(\frac{\partial\bar\varphi_2}{\partial T}\right)^2 - \frac{\partial\bar\varphi_2}{\partial R}\frac{\partial\bar\varphi_1}{\partial R}\right\} = 0. \tag{63}$$

But this is precisely the condition (a_3), as is easily seen on calculating the requisite Riemannian symbols for the form (61).

In the exceptional case $\varphi_3 = \text{const.}$ no transformation is necessary; (a_3) reduces to $(10, 10) = 0$, that is, the Gaussian curvature of the 2-space $d\sigma^2 = -\varphi_1 dt^2 + \varphi_2 dr^2$ must be zero. This condition is not satisfied for the space (21′), φ_3 being a constant; it is therefore of the second class. Hence, the space (57) is also of the second class.

A necessary and sufficient condition that a 2-space (u, v) shall be of class zero is that it shall admit an abelian group $\partial f/\partial u$, $\partial f/\partial v$ as a group of motions. This is the group-property which characterizes all surfaces of zero Gaussian curvature. The condition (a_3) reduces, in the case of the transformed element (21‴), to the simpler form

$$(02, 02)(13, 13) = (01, 01)(23, 23),$$

which by (63) is equivalent to the condition that the sub-space $\varphi = \text{const.}$, $\theta = \pm iR + \text{const.}$, or the space (62), shall admit an abelian group G_2

of two parameters as a group of motions. Theorems VII and VIII may therefore be stated thus:

A necessary and sufficient condition that the space (21‴) *shall be of the first class is that the sub-space* (62) *shall admit an abelian group* G_2 *as a group of motions. A 4-space which admits the group* G_1, *i.e. the space* (21′), *is at most of the second class.*

A similar statement holds for spaces (21′) for which $\varphi_3 = \text{const.}$

14. Let the space be static and its linear element written in the form

$$-ds^2 = dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \varphi_2\, dR^2 + \varphi_1\, dT^2,$$

where $T = it$. The spread has the coördinates

$$(64)\qquad \begin{aligned} x_1 &= R\sin\theta\sin\varphi, \qquad x_2 = R\sin\theta\cos\varphi, \quad x_3 = R\cos\theta, \\ x_4 &= \int\sqrt{\left[\varphi_2 - \frac{k^2}{4}\frac{(\varphi_1')^2}{\varphi_1}\right]}\, dR, \quad x_5 = k\sqrt{\varphi_1}\cos\frac{T}{k}, \quad x_6 = k\sqrt{\varphi_1}\sin\frac{T}{k}. \end{aligned}$$

The auxiliary surface is a surface of revolution generated by revolving the curve

$$x_4 = \int\sqrt{\varphi_2 - \frac{k^2(\varphi_1')^2}{4\varphi_1}}\, dR, \qquad x_5 = k\sqrt{\varphi_1}$$

about the x_4-axis. The surface $x_1 = R\sin\varphi$, $x_2 = R\cos\varphi$, $x_4 = \sqrt{\varphi_2}\, dR$ is a generalization of Flamm's quartic surface.*

If in (64) we put $(\varphi_1')^2 = (4/k^2)\varphi_1\varphi_2$, $x_4 = 0$, and the linear element is of class 1. If we put $\varphi_2 = \alpha/(R-\alpha)$, $\varphi_1 = c^2(1-\alpha/R)$ and $k = 1$, we get the representation of the Einstein solar field in a flat 6-space which was obtained by Kasner in a slightly different form.†

15. Static spaces which are not centro-symmetric are in general of a class higher than 2. Thus *the general static space*

$$(65)\qquad -ds^2 = g_0 dx_0^2 + g_1 dx_1^2 + g_2 dx_2^2 + g_3 dx_3^2, \quad x_0 = it$$

* H. Weyl, loc. cit., p. 236.

† E. Kasner, *Finite representation of the solar gravitational field in a flat space of six dimensions*, American Journal of Mathematics, vol. 43, pp. 130–133. It follows as a corollary from Theorem VIII that a general space (21′) can be conformally represented on a flat 5-space, and, when (a_3) is satisfied, on a flat 4-space. The space of an Einstein solar field can therefore not be represented conformally on a euclidean 4-space.

is of the fifth class, or one less than the maximum. To prove this we shall write the element in the form

$$(65') \quad -ds^2 = g_0\left[dx_0^2 + \frac{g_1}{g_0}dx_1^2 + \frac{g_2}{g_0}dx_2^2 + \frac{g_3}{g_0}dx_3^2\right] = g_0(dx_0^2 + d\sigma^2).$$

But a general curved 3-space may be immersed in a euclidean 6-space so that we have

$$-ds^2 = g_0\left[dx_0^2 + \sum_1^6 dy_i^2\right],$$

the y's being functions of x_1, x_2, x_3. The space (65) is thus represented conformally on a euclidean 7-space. *Such a space may be immersed in a euclidean 9-space*; for, write

$$z_1 = \sqrt{g_0}\,x_0, \qquad z_2 = \sqrt{g_0}\,y_1, \cdots, z_7 = \sqrt{g_0}\,y_6,$$

$$z_8 + z_9 = -\sqrt{g_0}, \qquad z_8 - z_9 = \sqrt{g_0}\left(\sum_1^6 y_i^2 + x_0^2\right),$$

and an easy calculation shows that

$$-ds^2 = g_0\left(dx_0^2 + \sum_1^6 dy_i^2\right) = \sum_1^7 dz_i^2 + dz_8^2 - dz_9^2.$$

Since a 4-space which admits a group G_1 as complete group of motions can always be reduced to the form (65) it follows that *such a space is of the fifth class.*

If $\partial f/\partial x_0$ is a translation, g_0 is const. and (65) is of the third class. If, in (65'), the sub-space $d\sigma^2$ admits the group $\partial f/\partial x_3$, g_1/g_0, g_2/g_0, g_3/g_0 are functions of x_1 and x_2 alone. As will be proved in § 16, a 3-space of this kind is of the first class, i. e. it can be immersed in a euclidean 4-space. (65) is therefore of the third class.

16. Let the space (65) admit the abelian group $\partial f/\partial x_0$, $\partial f/\partial x_3$ as complete group of motions, in which case the g's are functions of x_1 and x_2 alone. We shall prove that (65) is of the third class. We write (65) in the form

$$-ds^2 = d\sigma^2 + g_3 dx_3^2;$$

$d\sigma^2$ is the line-element of a 3-spread in a euclidean 4-space. To prove this we calculate the Riemannian symbols, of which the following are non-vanishing:

$$(10, 10), \qquad (20, 20), \qquad (12, 12), \qquad (10, 20),$$

and set up the Gaussian equations for the calculation of the b's. We find

$$b_{00}\,b_{11} = (10,\,10), \quad b_{00}\,b_{22} = (20,\,20), \quad b_{00}\,b_{12} = (10,\,20),$$
$$b_{11}\,b_{22} - b_{12}^2 = (12,\,12).$$

These equations determine the b's and they also satisfy the four Codazzi equations, which we shall not take the trouble of writing here. The line-element is now

$$-ds^2 = dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 + g_3\,dx_3^2.$$

If we put

$$y_6 + y_7 = \sqrt{g_3}, \qquad y_6 - y_7 = -x_3^2\sqrt{g_3}, \qquad y_5 = \sqrt{g_3}\,x_3,$$

we have the final result

$$-ds^2 = \sum_1^6 dy_i^2 - dy_7^2. \tag{66}$$

If $\partial f/\partial x_3$ is a translation, g_3 is constant and the 4-spread belongs to a euclidean 5-space. Its class is 1.

A notable space of this kind is Weyl's cylindrical and static space

$$-ds^2 = h(dz^2 + dr^2) + \frac{r^2\,d\theta^2}{f} - f\,dt^2,$$

which is of the third class and admits the group $\partial f/\partial t$, $\partial f/\partial\theta$ as a complete group of motions.* If $hf = 1$ we have the static centro-symmetric space of class 2 which admits a G_4.

17. Let the 4-space admit the group $\partial f/\partial x_0$, $\partial f/\partial x_3$, $\partial f/\partial x_2$; it is found that no reduction in class takes place. The space is of the third class as in the preceding case.

It thus appears that the complete group of a 4-space (65) determines its class, at least in the case of the abelian group, the group G_3 of "rotations" and the group of "translations." Whether this is true for all the groups of motions in a 4-space is an open question that might be worth while answering. Fubini's classification of 4-spaces (vols. 8 and 9, Annali di Matematica) would here render a notable service. It should however be noted that the group of certain sub-spaces will also play a rôle in the determination of the class.

* H. Weyl, Annalen der Physik, vol. 54, pp. 134–137.

West Virginia University,
Morgantown, W. Va.

A GENERALIZATION OF LEVI-CIVITA'S PARALLELISM AND THE FRENET FORMULAS*

BY

JAMES HENRY TAYLOR

Introduction. In the Riemann geometry of n dimensions the arc length t of a curve is given by the value of an integral of the form

$$(1) \qquad t = \int_{u_1}^{u} \sqrt{g_{\alpha\beta}\, x'^{\alpha} x'^{\beta}}\, du,$$

where the coefficients $g_{\alpha\beta}$ are functions of $x^1, \cdots, x^n$ only, and where the notation implies that α and β are summed from 1 to n. The vector analysis of such a geometry has been rather systematically developed.† Levi-Civita has developed a theory of parallelism‡ for the Riemann space, in which a vector ξ defined at each point of a curve is said to remain parallel to itself as it moves along the curve if it satisfies the system of equations

$$(2) \qquad \frac{d\xi^{\lambda}}{du} + \{\alpha\beta, \lambda\}\, x'^{\beta} \xi^{\alpha} = 0 \qquad (\lambda = 1, \cdots, n),$$

where $\{\alpha\beta, \lambda\}$ is the Christoffel symbol of the second kind formed with respect to the coefficients $g_{\alpha\beta}$ occurring in the expression (1) for the arc length. He has shown that if ξ_1 and ξ_2 are two variable vectors which are defined for each point of a curve, and which satisfy equations (2), then the angle between the vectors remains constant as they move along the curve.

---

* Presented to the Society, April 19, 1924.

This paper is essentially as submitted for a thesis to The University of Chicago, and was prepared with the cooperation of Professor G. A. Bliss.

† Some of the more recent works treating of this subject are

D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, 1923, which also contains an excellent bibliography;

F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, 1922;

A. S. Eddington, *The Mathematical Theory of Relativity*, 1923;

G. Ricci et T. Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications*, Mathematische Annalen, vol. 54 (1901), pp. 125–201.

‡ T. Levi-Civita, *Nozione di parallelismo in una varietà qualunque et conseguente specificazione geometrica della curvatura Riemanniana*, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

In the present paper a more general space is considered in which the arc length is defined by the value of an integral of the form

$$(3)\qquad t = \int_{u_1}^{u} F(x^1, \cdots, x^n; x'^1, \cdots, x'^n)\, du.$$

The geometry of such a space has been considered by Finsler.*

The measurement of vectors and angles in a Riemann space is with respect to the matrix of coefficients $g_{\alpha\beta}$. In the more general space here considered the matrix used for this purpose is

$$f_{\alpha\beta} = \frac{\partial F}{\partial x'^\alpha}\frac{\partial F}{\partial x'^\beta} + F\frac{\partial^2 F}{\partial x'^\alpha \partial x'^\beta},$$

and hence in general the angle between two vectors depends upon a parameter direction, as well as upon the point in space at which the vectors are taken. An interesting geometric interpretation is given to this situation in terms of the "indicatrix" of the calculus of variations associated with the integral (3), and a quadratic manifold which osculates the indicatrix.

In the present paper a differentiation operation corresponding to the left member of (2) is developed, and by means of it many of the results of Levi-Civita are generalized for the more general space here under consideration.

Blaschke† by means of the parallelism of Levi-Civita has derived the Frenet formulas for a twisted curve in a Riemann space of n dimensions. By following a method analogous to that of Blaschke, and making use of the extensions of the notions of parallelism and the measurement of angles which are here developed, it is found that the Frenet formulas may be obtained for any space in which the arc length is given by (3).

1. **The functions F and f.** Let the equations of a curve C in an n-dimensional space be

$$x^\alpha = x^\alpha(u), \quad u_1 \leq u \leq u_2 \qquad (\alpha = 1, \cdots, n).$$

We suppose the space to be such that the arc length t of the curve is given by the value of an integral of the form

$$(3)\qquad t = \int_{u_1}^{u} F(x, x')\, du,$$

* P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, 1918.

† W. Blaschke, *Frenets Formeln für den Raum von Riemann*, Mathematische Zeitschrift, vol. 6 (1920), pp. 94–99.

where we have employed the vector notation x and x' to represent the vectors $(x^1, \cdots, x^n)$ and $(x'^1, \cdots, x'^n)$ respectively, and where the primes denote derivatives with respect to u. A necessary and sufficient condition that the value of the integral (3) shall be independent of the parametric representation of the curve along which the integral is taken is that F shall satisfy the homogeneity condition*

$$(4) \qquad F(x, \varkappa x') = \varkappa F(x, x'), \qquad \varkappa > 0.$$

We shall suppose the function F to be positive and to satisfy the condition (4) at every point of the region of space which we are considering. Furthermore it will be assumed that the quadratic form

$$F_{\alpha\beta}\, \xi^\alpha \xi^\beta > 0, \qquad \xi \nparallel x',$$

where the symbol $F_{\alpha\beta}$ defined by $F_{\alpha\beta} = \partial^2 F / \partial x'^\alpha \partial x'^\beta$ has been introduced for convenience of notation. Here α and β are summed from 1 to n, as will be always understood in the following whenever an index letter occurs twice in the same term. The curve C and the function F will be considered to be real and to possess such continuity properties as may be required in the subsequent development.

As a consequence of the homogeneity condition (4) the relations

$$(5) \qquad F_\alpha x'^\alpha = F, \qquad F_{\alpha\beta} x'^\alpha = 0$$

are identically satisfied.

Let a function $f(x, x')$ be introduced by the equation

$$F = \sqrt{2f}$$

where now $f = F^2/2$ satisfies the homogeneity condition

$$f(x, \varkappa x') = \varkappa^2 f(x, x'), \qquad \varkappa > 0,$$

and the resulting identities obtained by differentiation,

$$(6) \qquad \begin{aligned} &f_\alpha x'^\alpha = 2f, && f_{\alpha\beta} x'^\alpha = f_\beta, \\ &f_{\alpha\beta\gamma} x'^\alpha = 0, && f_{\alpha\beta} = F_\alpha F_\beta + F F_{\alpha\beta} = f_{\beta\alpha}. \end{aligned}$$

Here the subscripts applied to f indicate the partial derivatives with respect to the corresponding x' variables. We assume the determinant $f_{\alpha\beta}$ to be

---

* O. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 195.

different from zero and denote by $f^{\alpha\beta}$ the element of the reciprocal matrix corresponding to $f_{\alpha\beta}$. Then

$$f_{\alpha\beta}f^{\alpha\varepsilon} = \delta^{\varepsilon}_{\beta}, \tag{7}$$

where $\delta^{\beta}_{\beta}=1$ and $\delta^{\varepsilon}_{\beta}=0$ for $\beta \neq \varepsilon$. This assumption that the determinant $f_{\alpha\beta} \neq 0$ may be shown to be equivalent to supposing the F_1 function of the calculus of variations to be non-vanishing.*

Consider the "indicatrix" of the calculus of variations† associated with the integral (3), and which is defined to be the manifold in the space of the x' variables determined by the equation $F(x, x') = 1$, where we regard x as fixed. On account of (5) and (6) the equation of the indicatrix may be given the form

$$\sqrt{f_{\alpha\beta}(x, x')\, x'^{\alpha} x'^{\beta}} = 1 .$$

The equation

$$\sqrt{f_{\alpha\beta}(x, r)\, x'^{\alpha} x'^{\beta}} = 1$$

for a fixed x and r defines another manifold in the x' space, which may well be called the *osculating indicatrix*, since it has contact of the second order with the original indicatrix at an arbitrary point $x' = r$ on it.‡ In the Riemann geometry these two manifolds are clearly coincident.

2. **Tensors and invariants.** Let the equations

$$\begin{aligned} y^i &= y^i(x^1, \cdots, x^n), & y'^i &= \frac{\partial y^i}{\partial x^{\alpha}} x'^{\alpha}, \\ x^{\alpha} &= x^{\alpha}(y^1, \cdots, y^n), & x'^{\alpha} &= \frac{\partial x^{\alpha}}{\partial y^i} y'^i, \end{aligned} \tag{8}$$

where the primes denote derivatives with respect to u, define a regular extended point transformation T and its inverse. We record here for future reference some relations obtained from (8):

* For a definition of the F_1 function and some of its properties, see M. Mason and G. A. Bliss, *The properties of curves in space which minimize a definite integral*, these Transactions, vol. 9 (1908), p. 441.

For a proof of the statement made above, see J. H. Taylor, *Reduction of Euler's equations to a canonical form*, Bulletin of the American Mathematical Society, vol. 31 (1925).

† C. Carathéodory, *Über die starken Maxima und Minima bei einfachen Integralen*, Mathematische Annalen, vol. 62 (1906), p. 456; also Bolza, loc. cit., p. 247.

‡ P. Finsler, loc. cit., p. 42.

$$
(9)\qquad
\begin{aligned}
y''^i &= \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} x'^\alpha x'^\beta + \frac{\partial y^i}{\partial x^\alpha} x''^\alpha;\\
x''^\alpha &= \frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} y'^i y'^k + \frac{\partial x^\alpha}{\partial y^i} y''^i;
\end{aligned}
$$

$$
(10)\qquad \frac{\partial y'^i}{\partial x^\beta} = \frac{\partial^2 y^i}{\partial x^\beta \partial x^\alpha} x'^\alpha, \qquad \frac{\partial x'^\alpha}{\partial y^k} = \frac{\partial^2 x^\alpha}{\partial y^k \partial y^i} y'^i;
$$

$$
(11)\qquad \frac{\partial y'^i}{\partial x'^\beta} = \frac{\partial y^i}{\partial x^\beta}, \qquad \frac{\partial x'^\alpha}{\partial y'^k} = \frac{\partial x^\alpha}{\partial y^k}.
$$

In most of the literature dealing with the tensor analysis the tensor components or coefficients are considered to be point functions. In the present paper, however, the tensor components will be allowed to be functions of $x^1, \cdots, x^n$ and their derivatives with respect to a scalar variable u.* With this extension in mind we adopt the formal definitions of tensors which are given in the books dealing with the subject.† Any set of n quantities $X^\alpha(x, x', x'', \cdots)$ ($\alpha = 1, \cdots, n$) which transform by the extended transformation T into n new quantities $Y^i(y, y', y'', \cdots)$ in such a way that

$$
Y^i = X^\alpha \frac{\partial y^i}{\partial x^\alpha}
$$

will be called a *contravariant tensor of rank* 1. Of course the substitution from one system of coördinates to the other must be completely carried out by adjoining to (8) and (9) corresponding relations involving higher derivatives if necessary. A *covariant tensor of rank* 1 is a set of n quantities X_α which transform by T into

$$
Y_i = X_\alpha \frac{\partial x^\alpha}{\partial y^i}.
$$

If a set of n^3 quantities $X^\alpha{}_{\beta\gamma}$ transform by T into

$$
Y^i{}_{jk} = X^\alpha{}_{\beta\gamma} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k},
$$

they are said to constitute a mixed tensor, contravariant of rank 1 and covariant of rank 2. The extension of the definitions to tensors of any rank is immediate.

* For some examples of tensors of this kind see F. D. Murnaghan, loc. cit., p. 88.

† D. J. Struik, loc. cit., p. 17; F. D. Murnaghan, loc. cit., p. 17; A. S. Eddington, loc. cit., pp. 51–52.

Denote by $h(y, y')$ the result of transforming $f(x, x')$. Then

$$h(y, y') = f(x, x')$$

and hence

$$\frac{\partial h}{\partial y'^i} = \frac{\partial f}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial y'^i} = \frac{\partial f}{\partial x'^\alpha}\frac{\partial x^\alpha}{\partial y^i}$$

by (11), that is, $\partial f/\partial x'^\alpha$ is a covariant tensor of rank 1.* A second differentiation gives

$$\frac{\partial^2 h}{\partial y'^i\,\partial y'^j} = \frac{\partial^2 f}{\partial x'^\alpha\,\partial x'^\beta}\frac{\partial x'^\beta}{\partial y'^j}\frac{\partial x^\alpha}{\partial y^i}$$
$$= \frac{\partial^2 f}{\partial x'^\alpha\,\partial x'^\beta}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\alpha}{\partial y^i}.$$

Therefore, $f_{\alpha\beta} = \partial^2 f/\partial x'^\alpha\,\partial x'^\beta$ is a covariant tensor of rank 2. Similarly $f_{\alpha\beta\gamma} = \partial f_{\alpha\beta}/\partial x'^\gamma$ is a covariant tensor of rank 3. From the fact that $f_{\alpha\beta}$ is a covariant tensor of rank 2 it follows that $f^{\alpha\beta}$ is a contravariant tensor of rank 2.†

We define the *inner product* of two contravariant vectors (tensors of rank 1) ξ_1 and ξ_2 to be the value of the bilinear form

$$f_{\alpha\beta}(x, r)\,\xi_1^\alpha\,\xi_2^\beta = (\xi_1, \xi_2)_r = (\xi_2, \xi_1)_r.$$

The value of this form is an *invariant* under the transformation T.‡

Note that in general the inner product of two vectors depends upon a parameter direction vector r; in case it does not it can be shown that the space is Riemannian.§ It will be observed, however, that the value of $(\xi_1, \xi_2)_r$ is independent of the magnitude of r since $f_{\alpha\beta}$ is homogeneous of degree zero in x'. If the directions ξ_1 and ξ_2 are such that $(\xi_1, \xi_2)_r = 0$, they will be said to be *orthogonal-r*. The value of the quadratic form

$$(\xi, \xi)_r = f_{\alpha\beta}\,\xi^\alpha\,\xi^\beta$$

will be taken as the *r-norm* of ξ. To justify this terminology it will be desirable to show that the value of the form $(\xi, \xi)_r$ is always positive

* F. D. Murnaghan, loc. cit., p. 88.

† F. D. Murnaghan, loc. cit., p. 42.

‡ F. D. Murnaghan, loc. cit., p. 40; A. S. Eddington, loc. cit., p. 53.

§ See P. Finsler, loc. cit., p. 40.

and greater than zero unless $\xi = (0, \cdots, 0)$, in which case it is obviously zero. Expressing $f_{\alpha\beta}$ in terms of F we have

$$(\xi, \xi)_r = (F_\alpha \xi^\alpha)^2 + F F_{\alpha\beta} \xi^\alpha \xi^\beta,$$

the arguments in F being x and r. Under the hypotheses of § 1 these two terms cannot vanish simultaneously. For, the first term is zero only when ξ is *transversal* to r.* Moreover it follows from the condition (5) that a direction transversal to r is different from r. The second term in the expression for $(\xi, \xi)_r$ vanishes only for $\xi = r$, in which case $(F_\alpha \xi^\alpha)^2$ becomes $F^2 > 0$ by (5). A vector whose norm is 1 will be said to be *unitary*.

It will be noticed that these invariants as here defined are not associated directly with the indicatrix as are the corresponding ones in the case of the Riemann geometry; they do, however, bear a similar relation to the osculating indicatrix. In the case of two orthogonal-r directions the situation may be characterized as follows: If ξ_1 and ξ_2 are orthogonal-r, they are conjugate directions in the sense of analytic geometry, not with respect to the indicatrix, but with respect to an osculating indicatrix which is uniquely determined by r.

3. **Generalization of Levi-Civita's parallelism.** Let X be a contravariant tensor of rank 1,

$$Y^m = X^\alpha \frac{\partial y^m}{\partial x^\alpha}.$$

Differentiating with respect to u we obtain

$$\frac{dY^m}{du} = \frac{dX^\alpha}{du} \frac{\partial y^m}{\partial x^\alpha} + X^\alpha \frac{\partial^2 y^m}{\partial x^\alpha \partial x^\beta} x'^\beta. \tag{12}$$

In the case of the Riemann geometry, the elimination of $\partial^2 y^m / \partial x^\alpha \partial x^\beta$ from the last term by means of the Christoffel transformation equations† leads to a contravariantive expression,

$$\frac{dX^\alpha}{du} + \{\lambda \beta, \alpha\} x'^\beta X^\lambda, \tag{13}$$

which forms the basis of many of the results of the paper by Levi-Civita on parallelism,‡ and is the differentiation process used by Blaschke in obtaining the Frenet formulas for a Riemann space.§ In the more general

* O. Bolza, loc. cit., p. 303; P. Finsler, loc. cit., p. 36.

† A. S. Eddington, loc. cit., p. 66; F. D. Murnaghan, loc. cit., p. 92.

‡ T. Levi-Civita, loc. cit.

§ W. Blaschke, loc. cit.

space under consideration here it is possible to make a desirable substitution for the coefficient of X^α in (12) which will give a generalization of the expression (13) having many of its properties.

The Christoffel three-index symbols are defined as follows:*

$$[\alpha\beta,\lambda] = \frac{1}{2}\left(\frac{\partial f_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial f_{\beta\lambda}}{\partial x^\alpha} - \frac{\partial f_{\alpha\beta}}{\partial x^\lambda}\right) = [\beta\alpha,\lambda], \tag{14}$$

$$\{\alpha\beta,\lambda\} = \Gamma_{\alpha\beta}^{\ \ \lambda} = f^{\lambda\mu}[\alpha\beta,\mu] = \Gamma_{\beta\alpha}^{\ \ \lambda}. \tag{15}$$

They satisfy the relations

$$f_{\lambda\mu}\Gamma_{\alpha\beta}^{\ \ \lambda} = [\alpha\beta,\mu] \tag{16}$$

and

$$[\alpha\beta,\lambda]+[\lambda\beta,\alpha] = \frac{\partial f_{\alpha\lambda}}{\partial x^\beta}. \tag{17}$$

In the present instance these symbols are functions of x and x', as they are formed from the $f_{\alpha\beta}(x,x')$. In taking the partial derivatives with respect to x the x' variables are treated as constants.

We have seen that $f_{\alpha\beta}$ is a covariant tensor of rank 2, i. e.,

$$h_{ik} = f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}.$$

Let us differentiate this identity with respect to y^j, remembering that the right member is not only a function of y through x but also through x':

$$\begin{aligned}\frac{\partial h_{ik}}{\partial y^j} &= f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^i\,\partial y^j}\frac{\partial y^\beta}{\partial x^k} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial^2 x^\beta}{\partial y^k\,\partial y^j}\right)\\ &\quad + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\gamma}{\partial y^j}\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^j}f_{\alpha\beta\gamma'}.\end{aligned} \tag{18$_1$}$$

Now form the two similar expressions

$$\begin{aligned}\frac{\partial h_{jk}}{\partial y^i} &= f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^j\,\partial y^i}\frac{\partial x^\beta}{\partial y^k} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial^2 x^\beta}{\partial y^k\,\partial y^i}\right)\\ &\quad + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\gamma}{\partial y^i}\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^i}f_{\alpha\beta\gamma'},\end{aligned} \tag{18$_2$}$$

* L. Bianchi, *Lezioni de Geometria Differenziale*, 1902, vol. 1, pp. 64–65; F. D. Murnaghan, loc. cit., p. 89.

$$
(18_3)\qquad \frac{\partial h_{ij}}{\partial y_k} = f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^i\,\partial y^k}\,\frac{\partial x^\beta}{\partial y^j} + \frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial^2 x^\beta}{\partial y^j\,\partial y^k}\right) + \frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^j}\,\frac{\partial x^\gamma}{\partial y^k}\,\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^j}\,\frac{\partial x'^\gamma}{\partial y^k}\,f_{\alpha\beta\gamma}.
$$

One half the sum of the first two of these expressions minus the third is the Christoffel index symbol $[ij,k]^*$ for the y-coördinate system, and we find

$$
(19)\qquad \begin{aligned}[ij,k]^* = {} & f_{\alpha\beta}\,\frac{\partial^2 x^\alpha}{\partial y^i\,\partial y^j}\,\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta,\gamma]\,\frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^j}\,\frac{\partial x^\gamma}{\partial y^k} \\ & + \frac{1}{2}\left(\frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^k}\,\frac{\partial x'^\gamma}{\partial y^j}\,f_{\alpha\beta\gamma} + \frac{\partial x^\alpha}{\partial y^j}\,\frac{\partial x^\beta}{\partial y^k}\,\frac{\partial x'^\gamma}{\partial y^i}\,f_{\alpha\beta\gamma}\right. \\ & \left. - \frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^j}\,\frac{\partial x'^\gamma}{\partial y^k}\,f_{\alpha\beta\gamma}\right).\end{aligned}
$$

Multiply (19) by $h^{km}(\partial x^\lambda/\partial y^m)\,y'^j$ and obtain

$$
(20)\qquad \begin{aligned}\Gamma^{*m}_{ij}\,\frac{\partial x^\lambda}{\partial y^m}\,y'^j = {} & f_{\alpha\beta}\,\frac{\partial^2 x^\alpha}{\partial y^i\,\partial y^j}\,\frac{\partial x^\beta}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,y'^j \\ & + [\alpha\beta,\gamma]\,\frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^j}\,\frac{\partial x^\gamma}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,y'^j \\ & + \frac{1}{2}\,\frac{\partial x^\alpha}{\partial y^i}\,f_{\alpha\beta\gamma}\,\frac{\partial x^\beta}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,\frac{\partial x'^\gamma}{\partial y^j}\,y'^j \\ & + \frac{1}{2}\,\frac{\partial x'^\gamma}{\partial y^i}\,\frac{\partial x^\beta}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,f_{\alpha\beta\gamma}\,\frac{\partial x^\alpha}{\partial y^j}\,y'^j \\ & - \frac{1}{2}\,\frac{\partial x^\alpha}{\partial y^i}\,\frac{\partial x'^\gamma}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,f_{\alpha\beta\gamma}\,\frac{\partial x^\beta}{\partial y^j}\,y'^j.\end{aligned}
$$

where the terms have been grouped to facilitate the next reduction. The last two terms of the right hand expression reduce to zero by the third of (6). Remembering the contravariant properties of h^{km}, and substituting for $\partial x'^\gamma/\partial y^j$ from (10), it is seen that the above equation reduces to

$$
\begin{aligned}\Gamma^{*m}_{ij}\,\frac{\partial x^\lambda}{\partial y^m}\,y'^j = {} & f_{\alpha\beta}\,f^{\beta\lambda}\,\frac{\partial^2 x^\alpha}{\partial y^i\,\partial y^j}\,y'^j + [\alpha\beta,\gamma]\,f^{\gamma\lambda}\,\frac{\partial x^\alpha}{\partial y^i}\,x'^\beta \\ & + \frac{1}{2}\,\frac{\partial x^\alpha}{\partial y^i}\,f_{\alpha\beta\gamma}\,\frac{\partial x^\beta}{\partial y^k}\,h^{km}\,\frac{\partial x^\lambda}{\partial y^m}\,\frac{\partial^2 x^\gamma}{\partial y^i\,\partial y^r}\,y'^r\,y'^j.\end{aligned}
$$

By (7), (15) and (9) this becomes

$$\Gamma_{ij}^{*m} \frac{\partial x^\lambda}{\partial y^m} y'^j = \frac{\partial^2 x^\lambda}{\partial y^i \partial y^j} y'^j + \Gamma_{\alpha\beta}{}^{\lambda} \frac{\partial x^\alpha}{\partial y^i} x'^\beta$$
$$+ \frac{1}{2} \frac{\partial x^\alpha}{\partial y^i} f_{\alpha\beta\gamma} \frac{\partial x^\beta}{\partial y^k} h^{km} \frac{\partial x^\lambda}{\partial y^m} \left(x''^\gamma - \frac{\partial x^\gamma}{\partial y^j} y''^j \right),$$

and hence

$$\left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) \frac{\partial x^\lambda}{\partial y^m} \tag{21}$$
$$= \frac{\partial^2 x^\lambda}{\partial y^i \partial y^j} y'^j + \left(\Gamma_{\alpha\beta}{}^{\lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda} \right) \frac{\partial x^\alpha}{\partial y^i},$$

where the covariant property of $f_{\alpha\beta\gamma}$ has been made use of, and where the quantities in the parentheses have been made symmetric by an interchange of j and k in the second term of the left hand expression. Interchanging x and y with a corresponding change in the index letters gives

$$\left(\Gamma_{\alpha\beta}{}^{\lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda} \right) \frac{\partial y^m}{\partial x^\lambda} \tag{22}$$
$$= \frac{\partial^2 y^m}{\partial x^\alpha \partial x^\beta} x'^\beta + \left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) \frac{\partial y^i}{\partial x^\alpha}.$$

By means of this relation (12) becomes

$$\frac{dY^m}{du} + \left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) Y^i$$
$$= \left[\frac{dX^\alpha}{du} + \left(\Gamma_{\lambda\beta}{}^{\alpha} x'^\beta + \frac{1}{2} x''^\gamma f_{\lambda\beta\gamma} f^{\beta\alpha} \right) X^\lambda \right] \frac{\partial y^m}{\partial x^\alpha}.$$

Hence we have the conclusion

If X^α is a contravariant tensor of rank 1, *then θX^α is also a contravariant tensor of rank* 1, *the θ-process being defined by the equation*

$$\theta X^\alpha = \frac{dX^\alpha}{du} + \left(\Gamma_{\lambda\beta}{}^{\alpha} x'^\beta + \frac{1}{2} x''^\gamma f_{\lambda\beta\gamma} f^{\beta\alpha} \right) X^\lambda. \tag{23}$$

An analogous differentiation process may readily be established for a covariant vector. For, let X_α be a covariant tensor of rank 1. Then

$$Y_i = X_\alpha \frac{\partial x^\alpha}{\partial y^i},$$

from which

$$\frac{dY_i}{du} = \frac{dX_\alpha}{du}\frac{\partial x^\alpha}{\partial y^i} + X_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j.$$

Substituting from (21) this reduces to

$$\frac{dY_i}{du} - \left(\Gamma^{*\,m}_{ij} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm}\right) Y_m$$
$$= \left[\frac{dX_\alpha}{du} - \left(\Gamma_{\alpha\beta}^{\ \ \lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda}\right) X_\lambda\right] \frac{\partial x^\alpha}{\partial y^i}.$$

We therefore have the theorem

If X_α is a covariant tensor of rank 1, then θX_α is also a covariant tensor of rank 1, the θ-process in this case being defined by

$$(24) \qquad \theta X_\alpha = \frac{dX_\alpha}{du} - \left(\Gamma_{\alpha\beta}^{\ \ \lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda}\right) X_\lambda.$$

Let us consider one more special case, that of a covariant tensor of rank 2,

$$Y_{ij} = X_{\mu\nu} \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}.$$

By differentiation we obtain

$$\frac{dY_{ij}}{du} = \frac{dX_{\mu\nu}}{du}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} + X_{\mu\nu}\frac{\partial^2 x^\mu}{\partial y^i \partial y^k} y'^k \frac{\partial x^\nu}{\partial y^j} + X_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial^2 x^\nu}{\partial y^j \partial y^k} y'^k.$$

By means of (21) this equation becomes

$$\frac{dY_{ij}}{du} = \frac{dX_{\mu\nu}}{du}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}$$
$$+ X_{\mu\nu}\left[\left(\Gamma^{*\,m}_{ik} y'^k + \frac{1}{2} y''^r h_{ikr} h^{km}\right)\frac{\partial x^\mu}{\partial y^m}\right.$$
$$\left. - \left(\Gamma_{\alpha\beta}^{\ \ \mu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\frac{\partial x^\alpha}{\partial y^i}\right]\frac{\partial x^\nu}{\partial y^j}$$
$$+ X_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\left[\left(\Gamma^{*\,m}_{jk} y'^k + \frac{1}{2} y''^r h_{jkr} h^{km}\right)\frac{\partial x^\nu}{\partial y^m}\right.$$
$$\left. - \left(\Gamma_{\alpha\beta}^{\ \ \nu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\frac{\partial x^\alpha}{\partial y^j}\right],$$

and hence

$$\frac{dY_{ij}}{du} - \left(\Gamma^{*\ m}_{ik}\, y'^k + \frac{1}{2}\, y''^r\, h_{ikr}\, h^{km}\right) Y_{mj} - \left(\Gamma^{*\ m}_{jk}\, y'^k + \frac{1}{2}\, y''^r\, h_{jkr}\, h^{km}\right) Y_{im}$$
$$= \left[\frac{dX_{\mu\nu}}{du} - \left(\Gamma_{\mu\beta}^{\ \ \alpha}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\mu\beta\gamma}\, f^{\beta\alpha}\right) X_{\alpha\nu}\right.$$
$$\left. - \left(\Gamma_{\nu\beta}^{\ \ \alpha}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\nu\beta\gamma}\, f^{\beta\alpha}\right) X_{\mu\alpha}\right] \frac{\partial x^\mu}{\partial y^i}\, \frac{\partial x^\nu}{\partial y^j},$$

which shows that the quantity in the brackets on the right is a covariant tensor of rank 2.

Evidently the differentiation process θ applies to a tensor of any type and rank and yields a tensor of the same type and rank. The general rule may be formulated as follows:

To find the result of the θ-operation with respect to a curve

$$x^\alpha = x^\alpha(u) \qquad\qquad (\alpha = 1, \cdots, n)$$

when applied to a tensor $X^{\cdots\cdots}_{\cdots\cdots}$ *of any type, form the derivative*

$$\frac{dX^{\cdots\cdots}_{\cdots\cdots}}{du}$$

and for each contravariant index μ, $X^{\cdots\mu\cdot}_{\cdots\cdots}$, *add*

$$\left(\Gamma_{\alpha\beta}^{\ \ \mu}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\alpha\beta\gamma}\, f^{\beta\mu}\right) X^{\cdots\alpha\cdot}_{\cdots\cdots},$$

and for each covariant index μ, $X^{\cdots\cdots}_{\cdots\mu\cdot}$, *subtract*

$$\left(\Gamma_{\mu\beta}^{\ \ \alpha}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\mu\beta\gamma}\, f^{\beta\alpha}\right) X^{\cdots\cdots}_{\cdots\alpha\cdot}.$$

We now consider the application of the differentiation process to a product. Suppose ξ and η are each covariant vectors. Then

$$X_{\mu\nu} = \xi_\mu\, \eta_\nu$$

is a covariant tensor of rank 2. Consider the expression

$$(\theta\,\xi_\mu)\,\eta_\nu + \xi_\mu\,(\theta\,\eta_\nu) = \frac{d\xi_\mu}{du}\,\eta_\nu + \xi_\mu\,\frac{d\eta_\nu}{du} - \left(\Gamma_{\mu\beta}^{\ \ \alpha}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\mu\beta\gamma}\, f^{\beta\alpha}\right)\xi_\alpha\,\eta_\nu$$
$$- \left(\Gamma_{\nu\beta}^{\ \ \alpha}\, x'^\beta + \frac{1}{2}\, x''^\gamma\, f_{\nu\beta\gamma}\, f^{\beta\alpha}\right)\xi_\mu\,\eta_\alpha$$
$$= \theta\, X_{\mu\nu}.$$

*Therefore, the distributive rule for the θ-process applied to a product of the form $X_{\mu\nu} = \xi_\mu \eta_\nu$ holds as for ordinary differentiation.**

Consider the mixed tensor $\varphi_\mu{}^\nu$. Then

$$\theta \varphi_\mu{}^\nu = \frac{d\varphi_\mu{}^\nu}{du} - \left(\Gamma_{\mu\beta}{}^\alpha x'^\beta + \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha}\right)\varphi_\alpha{}^\nu + \left(\Gamma_{\alpha\beta}{}^\nu x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\nu}\right)\varphi_\mu{}^\alpha. \tag{25}$$

If we *contract* this tensor in the usual way by equating the upper and lower indices we obtain a scalar invariant function $S = \varphi_\mu{}^\mu$. The expression (25) then becomes

$$\theta S = \theta \varphi_\mu{}^\mu = \frac{dS}{du} + \left[\begin{array}{l} -\left(\Gamma_{\alpha\beta}{}^\mu x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right) \\ +\left(\Gamma_{\alpha\beta}{}^\mu x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\end{array}\right]\varphi_\mu{}^\alpha = \frac{dS}{du}.$$

Hence we have the theorem

If S is a scalar function expressed as a contraction of a tensor, $S = \varphi_\mu{}^\mu$, then θS is the same as the ordinary derivative of S with respect to u.

Since a product of tensors is a tensor, this theorem is true for any scalar function which is expressed as a contraction of tensors.

In the Riemann geometry the covariant derivative of the fundamental tensor $g_{\alpha\beta}$ is zero, and we shall now show that this property generalizes:

$$\begin{aligned} \theta f_{\mu\nu} &= \frac{df_{\mu\nu}}{du} - \left(\Gamma_{\mu\beta}{}^\alpha x'^\beta + \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha}\right) f_{\alpha\nu} \\ &\qquad - \left(\Gamma_{\nu\beta}{}^\alpha x'^\beta + \frac{1}{2} x''^\gamma f_{\nu\beta\gamma} f^{\beta\alpha}\right) f_{\mu\alpha} \\ &= \frac{\partial f_{\mu\nu}}{\partial x^\beta} x'^\beta - \Gamma_{\mu\beta}{}^\alpha x'^\beta f_{\alpha\nu} - \Gamma_{\nu\beta}{}^\alpha x'^\beta f_{\mu\alpha} \\ &\qquad + f_{\mu\nu\gamma} x''^\gamma - \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha} f_{\alpha\nu} - \frac{1}{2} x''^\gamma f_{\nu\beta\gamma} f^{\beta\alpha} f_{\mu\alpha}. \end{aligned}$$

By means of (16) and (7) this may be written

$$\begin{aligned} \theta f_{\mu\nu} &= \left(\frac{\partial f_{\mu\nu}}{\partial x^\beta} - [\mu\beta, \nu] - [\nu\beta, \mu]\right) x'^\beta + \left(f_{\mu\nu\gamma} - \frac{1}{2} f_{\mu\nu\gamma} - \frac{1}{2} f_{\nu\mu\gamma}\right) x''^\gamma \\ &= 0, \end{aligned}$$

by (17).

* This theorem has been established here for two tensors both covariant but it is easily seen to be true for the product of tensors of any type.

That is, the θ-derivative of the covariant tensor $f_{\alpha\beta}$ is zero.

A direction X^α defined along a curve C in a Riemann space remains *parallel* in the Levi-Civita sense* if it satisfies the system of linear differential equations obtained by equating to zero the expression (13). The θ-process here developed is clearly a generalization of (13) since it reduces to the latter if $f_{\alpha\beta}$ are merely point functions, i. e., in case the space is Riemannian. A theorem by Levi-Civita† generalizes at once.

Let ξ_1 and ξ_2 be two contravariant vectors which are defined at each point of the curve C as functions of x and its derivatives with respect to u. Moreover, suppose ξ_1 and ξ_2 satisfy the system of linear differential equations

$$\theta\,\xi^\alpha = 0 \qquad (\alpha = 1, \cdots, n) \tag{26}$$

which are associated with the curve. Then if the vectors are measured with respect to the tangent direction of the curve C, the norms of ξ_1 and ξ_2 remain constant, and the angle between ξ_1 and ξ_2 remains constant as the vectors move along the curve.

For each of the expressions

$$f_{\alpha\beta}\,\xi_1^\alpha\,\xi_1^\beta, \qquad f_{\alpha\beta}\,\xi_2^\alpha\,\xi_2^\beta, \qquad f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta$$

is a scalar function formed by a contraction of tensors, and therefore, considering the last one,

$$\begin{aligned} \frac{d}{du}(f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta) &= \theta(f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta) \\ &= (\theta f_{\alpha\beta})\,\xi_1^\alpha\,\xi_2^\beta + f_{\alpha\beta}(\theta\,\xi_1^\alpha)\,\xi_2^\beta + f_{\alpha\beta}\,\xi_1^\alpha(\theta\,\xi_2^\beta) \\ &= 0, \end{aligned}$$

since $\theta f_{\alpha\beta} = 0$, and ξ_1 and ξ_2 satisfy (26). Hence, each of the three above expressions is a constant, which establishes the theorem.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be a fundamental set of solutions of the system (26), and suppose $\eta_1, \eta_2, \cdots, \eta_n$ to be a linearly independent set of vectors expressed linearly in terms of $\xi_1, \cdots, \xi_n$ and with the additional property that $\eta_1, \cdots, \eta_n$ constitute a unitary orthogonal set.‡ Then as an immediate consequence of the theorem just given it follows that *the set $\eta_1, \cdots, \eta_n$ being initially a unitary orthogonal one remains unitary orthogonal all along the curve.*

* T. Levi-Civita, loc. cit.

† T. Levi-Civita, loc. cit., p. 182.

‡ A method of determining a unitary orthogonal system is discussed in the next section.

4. Orthogonalization process. Frenet formulas.* Let the equations of a curve C be

$$x^\alpha = x^\alpha(t), \qquad t_1 \leqq t \leqq t_2 \qquad (\alpha = 1, \cdots, n),$$

where now the curve is referred to the arc length t in the sense of (3), as the parameter. Such a selection of the parameter is always possible. For, if the equations of the curve are given in terms of an arbitrary parameter u it follows from the definition of the arc length

$$t = \int_{u_1}^{u} F\left(x, \frac{dx}{du}\right) du$$

that

$$F\left(x, \frac{dx}{dt}\right) = 1$$

is a necessary and sufficient condition that the independent variable be the arc length. Clearly this condition can always be satisfied by virtue of the homogeneous property of F. We shall assume for the remainder of this paper that such a choice of the independent variable has been made, and hereafter the primes will denote derivatives with respect to the arc length t.

As a consequence of the choice of the arc length as parameter it follows that the tangent vector $x' = dx/dt$ is a unitary vector, for by (5)

$$f_{\alpha\beta}(x, x')x'^\alpha x'^\beta = F^2(x, x').$$

The tangent vector x', which we denote hereafter by ξ_1, is a contravariant tensor of rank 1. Then $\xi_2 = \theta\xi_1$ is also a contravariant vector; likewise $\xi_3 = \theta\xi_2, \cdots,$ where the θ-operation is defined by (23) with arc length as the independent variable. Hence we can associate with each point of the curve a system of contravariant vectors, $\xi_1, \xi_2, \cdots, \xi_n$ which are obtained sequentially by repeated application of the directional derivative θ, thus

$$\xi_1 = x', \qquad \xi_k = \theta\xi_{k-1} \qquad (k = 2, 3, \cdots, n).$$

We suppose this system of vectors to be linearly independent so that they will form a basis for the whole vector space at the point of the curve C at which they are taken. Our first problem is to replace $\xi_1, \cdots, \xi_p$ $(p = 1, \cdots, n)$ by a set of vectors $\eta_1, \cdots, \eta_p$ which are equivalent in the

* This section follows very closely the paper by W. Blaschke, loc. cit.

sense that they define the same space, and which shall have the additional property that they constitute a *unitary orthogonal* system, which is defined by

$$(\eta_i, \eta_k)_{x'} = f_{\alpha\beta}(x, x')\,\eta_i^\alpha \eta_k^\beta = \delta_{ik},$$

where $\delta_{ii} = 1$ and $\delta_{ik} = 0$ for $i \neq k$. It is essential for use in the later development that this norming and orthogonalizing shall be with respect to the tangent direction x' as is indicated by the notation.

For simplicity of notation we denote the inner product, $(\xi_p, \xi_q)_{x'}$, by (p, q). We now define a set of vectors by the equations

$$\zeta_1 = \xi_1,\ \zeta_p = \begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi_1 \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p,p-1) & \xi_p \end{vmatrix} \qquad (p = 2, \cdots, n).$$

Then $f_{\alpha\beta}\,\zeta_p^\alpha \zeta_q^\beta = 0$ for $q = 1, 2, \cdots, p-1$, i. e., for $p > q$. The vectors $\zeta_1, \cdots, \zeta_n$ then constitute an orthogonal system; to make them into a unitary system it will only be necessary to divide each vector by the square root of its norm. Now

$$f_{\alpha\beta}\,\zeta_p^\alpha \zeta_p^\beta = D_{p-1}\,(f_{\alpha\beta}\,\xi_p^\alpha \zeta_p^\beta) = D_{p-1}\,D_p,$$

where D_p is defined by

$$D_0 = 1,\ D_p = \begin{vmatrix} (1,1) & \cdots & (1,p) \\ \cdot & \cdot & \cdot \\ (p,1) & \cdots & (p,p) \end{vmatrix}, \qquad (p = 1, \cdots, n). \tag{27}$$

Since the norm of $\zeta_p > 0$, it follows that $D_p > 0$. Hence the system of vectors $\eta_1, \cdots, \eta_n$ given by

$$\eta_1 = \xi_1,\ \eta_p = \frac{1}{\sqrt{D_{p-1}\,D_p}} \begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi_1 \\ \cdot & \cdot & \cdot & \cdot \\ (p,1) & \cdots & (p,p-1) & \xi_p \end{vmatrix} \quad (p = 2, \cdots, n) \tag{28}$$

is the unitary orthogonal system desired. Clearly they are linearly independent and form a basis for the whole vector space at the point of the curve C at which they are taken.

The vector η_p being a linear combination, with scalar coefficients, of contravariant vectors is itself a contravariant vector. Then $\theta\,\eta_p$ is a contravariant vector and can therefore be expressed as a linear combination of $\eta_1, \cdots, \eta_n$ in the form

$$\theta\,\eta_p^\alpha = C_{pq}\,\eta_q^\alpha \qquad (q \text{ summed from 1 to } n). \tag{29}$$

where

$$C_{pq} = f_{\alpha\beta}\,(\theta\,\eta_p^\alpha)\,\eta_q^\beta,$$

the coefficients C_{pq} being scalars or *invariants*. Following Blaschke and others we call these invariants the *curvatures* of the curve.

From

$$f_{\alpha\beta}\,\eta_p^\alpha\,\eta_q^\beta = 0, \qquad p \neq q,$$

we obtain

$$\theta\,(f_{\alpha\beta}\,\eta_p^\alpha\,\eta_q^\beta) = f_{\alpha\beta}\,(\theta\,\eta_p^\alpha)\,\eta_q^\beta + f_{\alpha\beta}\,\eta_p^\alpha\,(\theta\,\eta_q^\beta) = 0,$$

where we have made use of the distributive law for the θ-process applied to a product, the fact that $\theta f_{\alpha\beta} = 0$, and the theorem that the θ-derivative for a scalar function expressed as a contraction of tensors yields the same result as the ordinary derivative. Hence

$$C_{pq} + C_{qp} = 0. \tag{30}$$

Now η_p is a linear combination of $\xi_1, \cdots, \xi_p$, from which it follows that $\theta\,\eta_p$ is a linear combination of $\xi_1, \cdots, \xi_{p+1}$ or of $\eta_1, \cdots, \eta_{p+1}$ only, and therefore

$$C_{pq} = 0 \text{ for } q > p+1. \tag{31}$$

As a consequence of (30) and (31) we see that the matrix of coefficients C_{pq} is of the form

$$\| C_{pq} \| = \left\| \begin{array}{ccccc} 0 & +\frac{1}{\varrho_1} & 0 & \cdots & 0 \\ -\frac{1}{\varrho_1} & 0 & +\frac{1}{\varrho_2} & \cdots & 0 \\ 0 & -\frac{1}{\varrho_2} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right\|$$

where

$$\frac{1}{\varrho_p} = f_{\alpha\beta}\,(\theta\,\eta_p^\alpha)\,\eta_{p+1}^\beta, \qquad 0 < p < n,$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0.$$

We now write (29) in the form

$$\theta\,\eta_p = -\frac{1}{\varrho_{p-1}}\,\eta_{p-1} + \frac{1}{\varrho_p}\,\eta_{p+1}, \qquad 0 < p < n, \tag{32}$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0.$$

These formulas are the analogues of the well known Frenet formulas associated with a twisted curve in space.

It is desirable to compute the curvatures $1/\varrho_p$ in terms of the θ-derivatives along the curve. Notice that

$$\begin{aligned}\theta(p,q) &= \theta(f_{\alpha\beta}\xi_p^\alpha \xi_q^\beta)\\ &= (\theta f_{\alpha\beta})\xi_p^\alpha\xi_q^\beta + f_{\alpha\beta}(\theta\xi_p^\alpha)\xi_q^\beta + f_{\alpha\beta}\xi_p^\alpha(\theta\xi_q^\beta)\\ &= (p+1,q)+(p,q+1)\end{aligned}$$

since $\theta f_{\alpha\beta} = 0$. From (28)

$$\theta\eta_p^\alpha = \left[\theta\left(\frac{1}{\sqrt{D_{p-1}D_p}}\right)\right]\sqrt{D_{p-1}D_p}\,\eta_p^\alpha$$

$$+\frac{1}{\sqrt{D_{p-1}D_p}}\left[\begin{vmatrix}(2,1) & (1,2) & \cdots & \xi_1^\alpha\\ (3,1) & (2,2) & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ (p+1,1) & \cdots & \cdots & \xi_p^\alpha\end{vmatrix} + \begin{vmatrix}(1,1) & (2,2) & \cdots & \xi_1^\alpha\\ (1,2) & (3,2) & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ (p,1) & (p+1,2) & \cdots & \xi_p^\alpha\end{vmatrix}\right.$$

$$\left. + \cdots + \begin{vmatrix}(1,1) & \cdots & (1,p-1) & \xi_2^\alpha\\ \cdots & \cdots & \cdots & \cdots\\ (p,1) & \cdots & (p,p-1) & \xi_{p+1}^\alpha\end{vmatrix}\right].$$

Hence

$$\begin{aligned}\frac{1}{\varrho_p} &= f_{\alpha\beta}(\theta\eta_p^\alpha)\eta_{p+1}^\beta\\ &= \frac{1}{\sqrt{D_{p-1}D_p}}\begin{vmatrix}(1,1) & \cdots & (1,p-1) & 0\\ \cdots & \cdots & \cdots & \cdots\\ (p-1,1) & \cdots & (p-1,p-1) & 0\\ (p,1) & \cdots & (p,p-1) & (\xi_{p+1},\eta_{p+1})\end{vmatrix}\\ &= \frac{1}{\sqrt{D_{p-1}D_p}}\cdot D_{p-1}\cdot\frac{1}{\sqrt{D_pD_{p+1}}}\cdot D_{p+1},\end{aligned}$$

and, therefore,

$$\frac{1}{\varrho_p} = \frac{\sqrt{D_{p-1}D_{p+1}}}{D_p}. \tag{33}$$

This theorem may be formulated as follows:

Let $x^\alpha = x^\alpha(t)$, $t_1 \leq t \leq t_2$ $(\alpha = 1,\cdots,n)$ be the equations of a curve referred to the arc length as parameter. Form the system of contravariant

vectors $\xi_1, \cdots, \xi_n$ *defined by* $\xi_1 = x'$, $\xi_k = \theta\,\xi_{k-1}$ $(k = 2, \cdots, n)$. *Let these vectors be normed and orthogonalized with respect to the tangent direction to the curve yielding a unitary orthogonal set* $\eta_1, \cdots, \eta_n$ *of principal directions associated with each point of the curve. Then the Frenet formulas may be written in the form*

$$\theta\,\eta_p = -\frac{1}{\varrho_{p-1}}\eta_{p-1} + \frac{1}{\varrho_p}\eta_{p+1}, \qquad 0 < p < n,$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0,$$

where the pth curvature $1/\varrho_p$ *is given by* (33).

NOTE: I regret to say that adequate reference has not here been made to the paper, *A generalization of the Riemannian line-element* by J. L. Synge, these Transactions, this volume, pp. 61–67. When I returned the proof sheets of my paper, I was aware that Mr. Synge had written a paper on the same general subject, but the scanty and indirect account I had of his paper did not indicate much overlapping. It was only upon publication of Mr. Synge's paper that an adequate account of his results became available to me. The same remarks apply to § 4 of my paper *Reduction of Euler's equations to a canonical form*, Bulletin of the American Mathematical Society, vol. 31 (1925).

UNIVERSITY OF CHICAGO,
CHICAGO, ILL.